

# A computational approach to the Thompson group $F$

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**ABSTRACT.** Let  $F$  denote the Thompson group with standard generators  $A = x_0$ ,  $B = x_1$ . It is a long standing open problem whether  $F$  is an amenable group. By a result of Kesten from 1959, amenability of  $F$  is equivalent to

$$(i) \quad \|I + A + B\| = 3$$

and to

$$(ii) \quad \|A + A^{-1} + B + B^{-1}\| = 4,$$

where in both cases the norm of an element in the group ring  $\mathbb{C}F$  is computed in  $B(\ell^2(F))$  via the regular representation of  $F$ . By extensive numerical computations, we obtain precise lower bounds for the norms in (i) and (ii), as well as good estimates of the spectral distributions of  $(I + A + B)^*(I + A + B)$  and of  $A + A^{-1} + B + B^{-1}$  with respect to the tracial state  $\tau$  on the group von Neumann Algebra  $L(F)$ . Our computational results suggest, that

$$\|I + A + B\| \approx 2.95 \quad \|A + A^{-1} + B + B^{-1}\| \approx 3.87.$$

It is however hard to obtain precise upper bounds for the norms, and our methods cannot be used to prove non-amenability of  $F$ .

## 1. INTRODUCTION

**Definition 1.1.** The Thompson group  $F$  is the group of homeomorphisms  $g : [0, 1] \rightarrow [0, 1]$  for which:

- the endpoints satisfy  $g(0) = 0$ ,  $g(1) = 1$ ,
- it is piecewise linear with finitely many break points on the dyadic numbers  $\mathbb{Z}[\frac{1}{2}] \cap (0, 1)$ ,
- all slopes of  $g$  are in the set  $2^{\mathbb{Z}} := \{2^n \mid n \in \mathbb{Z}\}$ .

$F$  is a countable group, and it is generated by the elements  $A, B$  whose graphs are shown in Fig. 1.

Moreover, it has a finite presentation in terms of  $A, B$ , namely

$$F = \langle A, B \mid [AB^{-1}, A^{-1}BA] = [AB^{-1}, A^{-2}BA^2] = e \rangle,$$

where the group commutator  $[g, h] = ghg^{-1}h^{-1}$  is defined as usual.

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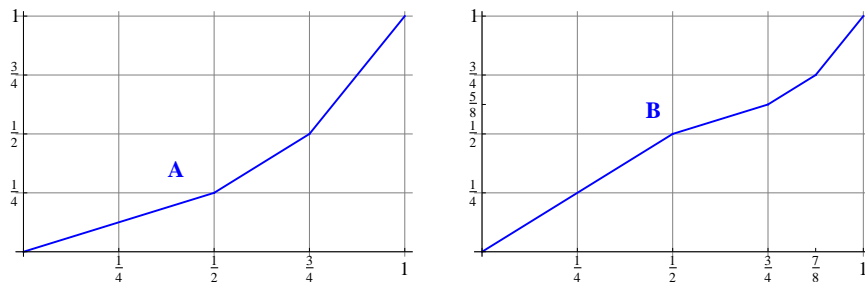
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FIGURE 1. The generators  $A, B$  of  $F$ .

Recall that elementary amenability implies amenability, and a copy of the free group  $\mathbb{F}_2$  (on two generators) inside a group implies non-amenability of the group. It is known that  $F$  is not elementary amenable, i.e.  $F$  cannot be obtained from finite or Abelian groups by taking subgroups, quotients, extensions, and direct limits. On the other hand, by a result of Brin and Squier [5],  $F$  does not contain a copy of  $\mathbb{F}_2$ . For more information on the Thompson group  $F$ , see the survey paper by Cannon, Floyd and Perry [8].

It is a main open problem to decide whether the Thompson group  $F$  is amenable. Recently, Monod [27] has constructed examples of groups of homeomorphisms of  $[0, 1]$  which are non-amenable, but which also do not contain a copy of  $\mathbb{F}_2$ . These groups resemble  $F$ . Moreover, Olesen and the second named author has shown in [19] that if the reduced  $C^*$ -algebra  $C_r^*(T)$  of the (non-amenable) Thompson group  $T$  is simple, then  $F$  is non-amenable. Both of the above mentioned results suggest that  $F$  might not be amenable, and extrapolations of our computational results point in the same (non-amenability) direction.

The present paper grew out of an attempt to test the amenability problem for  $F$  by using computers to estimate norms of certain elements in the group ring  $\mathbb{C}F$  of  $F$ . By the norm  $\|a\|$  (see also Section 2) of an element  $a$  in the group ring of a discrete group  $\Gamma$  we mean

$$(1) \quad \|a\| = \|\lambda(a)\|_{B(\ell^2(\Gamma))},$$

where  $\lambda$  is the left regular representation of  $\Gamma$ . As explained in Section 2, it is standard to write  $a \in B(\ell^2(\Gamma))$  instead of  $\lambda(a) \in B(\ell^2(\Gamma))$ , for any  $a \in \mathbb{C}F$ , and we will continue with this tradition. Our starting point is the following two theorems due to Kesten and Lehner: (See Section 7 for a more detailed discussion).

**Theorem 1.2** ([23],[25]). *Let  $\Gamma$  be a discrete group with a generating set  $X = \{s_1, \dots, s_k\}$  such that  $k \geq 2$  and  $e \notin X$ . Then,*

$$2\sqrt{k} \leq \|e + s_1 + \dots + s_k\| \leq k + 1.$$

*Moreover, the upper bound is attained if and only if  $\Gamma$  is amenable, and the lower bound is attained if and only if  $X$  generates  $\Gamma$  freely.*

**Theorem 1.3** ([23],[24]). *Let  $\Gamma$  be a discrete group with a generating set  $X = \{s_1, \dots, s_k\}$  such that  $k \geq 2$  and  $X \cap X^{-1} = \emptyset$ . Then,*

$$2\sqrt{2k-1} \leq \|s_1 + \dots + s_k + s_1^{-1} + \dots + s_k^{-1}\| \leq 2k.$$

Moreover, the upper bound is attained if and only if  $\Gamma$  is amenable, and the lower bound is attained if and only if  $X$  generates  $\Gamma$  freely.

Hence for the Thompson group  $F$  we get

**Corollary 1.4.** *Let  $A$  and  $B$  be the standard generators of  $F$ , and let  $I$  denote the unit element of  $F$ . Then*

$$2\sqrt{2} < \|I + A + B\| \leq 3,$$

$$2\sqrt{3} < \|A + A^{-1} + B + B^{-1}\| \leq 4.$$

Moreover, in both cases the upper bound is attained if and only if  $F$  is amenable.

Let  $L(\Gamma)$  denote the von Neumann algebra of a discrete group  $\Gamma$ , i.e.  $L(\Gamma)$  is the von Neumann algebra in  $B(\ell^2(\Gamma))$  generated by  $\lambda(\Gamma)$ . Then

$$(2) \quad \tau(T) = \langle T\delta_e, \delta_e \rangle, \quad T \in L(\Gamma)$$

defines a normal faithful tracial state on  $L(\Gamma)$  (See e.g. Section 6.7 in [22]). Hence

$$(3) \quad \|T\| = \|T^*T\|^{1/2} = \lim_{n \rightarrow \infty} \tau((T^*T)^n)^{\frac{1}{2n}}$$

(cf. Section 4). Hence if we knew all the numbers

$$m_n(T^*T) := \tau((T^*T)^n), \quad n \in \mathbb{N}_0$$

we could also compute the norm  $\|T\|$ . In practice we can only compute a finite number of the moments  $m_n(T^*T)$ .

In this paper, we develop efficient methods, both mathematically and computationally to compute the numbers  $m_n(T^*T)$  in the case

$$T = \sum_{x \in Y} x \in \mathbb{C}\Gamma$$

for any finite set  $Y$  in a discrete group  $\Gamma$ . We then apply the methods to the elements  $T_1$  and  $T_2$  in the group ring  $\mathbb{C}F$  of the Thompson group  $F$  given by

$$(4) \quad T_1 := I + A + B, \quad T_2 := A + A^{-1} + B + B^{-1}.$$

As a result, we have been able to compute the moments  $m_n(T_1^*T_1)$  for  $0 \leq n \leq 37$  and the moments  $m_n(T_2^*T_2) = m_{2n}(T_2)$  for  $0 \leq n \leq 24$ .

Using the spectral theorem to the self-adjoint operators

$$\tilde{T}_i := \begin{pmatrix} 0 & T_i^* \\ T_i & 0 \end{pmatrix} \in M_2(B(\ell^2(F))),$$

(cf. Section 2), one gets that there are unique probability measures  $\mu_1, \mu_2$  to  $\mathbb{R}$  with  $\text{supp}(\mu_i) \subset [-\|T_i\|, \|T_i\|]$  such that  $\mu_i$  is invariant under the reflection  $t \mapsto -t$ , and such that

$$(5) \quad \int_{-\|T_i\|}^{\|T_i\|} t^{2n} d\mu_i(t) = m_n(T_i^*T_i), \quad n \in \mathbb{N}_0, \quad i = 1, 2.$$

Moreover,  $\pm\|T_i\| \in \text{supp}(\mu_i)$ .

Using methods from the theory of orthonormal polynomials applied to these two measures (cf. Section 4) we obtain from our moment calculations good lower bounds for  $\|T_i\|$ ,  $i = 1, 2$ , namely

$$\|I + A + B\| \geq 2.86759$$

and

$$\|A + A^{-1} + B + B^{-1}\| \geq 3.60613.$$

In fact for each  $n \leq 37$  (resp.  $n \leq 24$ ), we find a lower estimate of  $\|T_1\|$  (resp.  $\|T_2\|$ ), and a suitable extrapolation of those two finite series of numbers suggests that the actual norms are much closer to 3 (resp 4), namely

$$\|I + A + B\| \approx 2.95 \quad \text{and} \quad \|A + A^{-1} + B + B^{-1}\| \approx 3.87.$$

Furthermore, based on our moment calculations, we have also been able to estimate the Lebesgue densities of the measures  $\mu_1$  and  $\mu_2$  with fairly high precision. This shows that the measures  $\mu_1$  and  $\mu_2$  are very close to zero on the interval  $[2.9, 3]$  and  $[3.7, 4]$  respectively, but we cannot rule out, that the measures have very “thin tails” stretching all the way up to 3 and 4, respectively, which would imply that  $F$  is amenable (cf. Corollary 1.4). In comparison, one gets for the free group  $\mathbb{F}_2$  on two generators  $a, b$  that

$$\|e + a + b\| = 2\sqrt{2} \approx 2.82824$$

and

$$\|a + a^{-1} + b + b^{-1}\| = 2\sqrt{3} \approx 3.46410.$$

The measures  $\mu_1, \mu_2$  based on  $a, b$  instead of  $A, B$ , will be denoted by  $\mu_i^{\text{free}}$  ( $i = 1, 2$ ), and they can be computed explicitly (See Section 7):

$$\begin{aligned} \mu_1^{\text{free}} &= \frac{3}{2\pi} \frac{\sqrt{8-x^2}}{9-x^2} 1_{[-2\sqrt{2}, 2\sqrt{2}]}(x) dx \\ \mu_2^{\text{free}} &= \frac{2}{\pi} \frac{\sqrt{12-x^2}}{16-x^2} 1_{[-2\sqrt{3}, 2\sqrt{3}]}(x) dx. \end{aligned}$$

Since 2007 a number of papers has been published about computational approaches to problems related to the Thompson group  $F$ , including the amenability problem (cf. [7],[3],[12],[13],[14]). Our paper is the first that considers the moments of  $T_1^* T_1$  for  $T_1 := I + A + B$ . In [7] Burillo, Cleary and Wiest use probabilistic methods to estimate the moments of  $m_n(T_2^* T_2) = m_{2n}(T_2)$  for  $T_2 := A + A^{-1} + B + B^{-1}$  for  $n = 10, 20, 30, \dots, 160$ . Moreover, Elder, Rechnitzer and Wong compute in [13] the first 22 cogrowth coefficients of  $F$  with respect to the symmetric set of generators  $\{A, A^{-1}, B, B^{-1}\}$ , from which the exact values of  $m_{2n}(T_2)$   $n = 1, \dots, 11$  can easily be computed. We will comment in more detail on the results of [7] and [13] at the end of Section 4.

The Thompson group  $F$  has been the key motivation for the research presented in this paper. However, we should stress, that the methods developed in this paper can be applied to do similar computations in any other finitely generated group, provided there is a reasonable fast algorithm to decide whether two words in the generators correspond to the same element of the group.

The rest of the paper is organized as follows: Section 2 contains background material on the reduced  $C^*$ -algebra  $C_r^*(\Gamma)$  and the group von Neumann algebra  $L(\Gamma)$  associated to a discrete group  $\Gamma$ . In Section 3, we consider for any given finite set  $Y$  in a discrete group  $\Gamma$ , the element

$$h = \sum_{s \in Y} s$$

and explain how one can compute the moments

$$m_n = \tau((h^* h)^n), \quad n \geq 1$$

from computing much smaller number  $\|h_n\|_2^2, \xi_n, \eta_n, \zeta_n$  ( $n \geq 1$ ) associated to the pair  $(\Gamma, Y)$ . We next apply this to the special case of  $\Gamma = F$ , the Thompson group  $F$ , and  $Y = \{I, A, B\}$  or  $Y = \{A, A^{-1}, B, B^{-1}\}$  in order to compute numerically the moments  $m_n$  for  $1 \leq n \leq 37$  in the first case and for  $n \leq n \leq 24$  for the second case. In Sections 4 and 5 we apply methods from the theory of orthogonal polynomials to estimate the norms  $\|I + A + B\|$  and  $\|A + A^{-1} + B + B^{-1}\|$ , and to estimate the probability measures  $\mu_1, \mu_2$ , associated to  $T_1 = I + A + B$  and  $T_2 = A + A^{-1} + B + B^{-1}$  via formula (5) above. The sections 6 and 7 are the main theoretical sections of the paper. In the general setting of a finite subset  $Y$  of a discrete group  $\Gamma$ , we derive in Section 6 the formulas (used in Section 3), that allow us to pass back and forth between the 5 sequences of numbers  $\|h_n\|_2^2, \xi_n, \eta_n, \zeta_n$  and  $m_n$  ( $n \geq 1$ ). In Section 7 we formulate and explain Kesten's and Lehner's results (Theorem 1.2 and Theorem 1.3) in the setting of Leinert sets (cf. Theorem 7.2). Moreover, we make comparison between our numbers  $\|h_n\|_2^2, \xi_n, \eta_n, \zeta_n$  ( $n \geq 1$ ) and the cogrowth coefficients due to Cohen [11] and Grigorchuk [16], and show that amenability of the group  $\Gamma_0 \subset \Gamma$  generated by  $Y^{-1}Y$  can be decided from the asymptotics of each of the 4 sequences mentioned above (cf. Corollary 7.9).

## 2. PRELIMINARIES

Let  $\Gamma$  be a countable discrete group. We consider the left regular representation  $\lambda$  of  $\Gamma$  on  $\ell^2(\Gamma)$  i.e.

$$(\lambda(x)f)(y) := f(x^{-1}y), \quad f \in \ell^2(\Gamma), \quad x, y \in \Gamma.$$

We will also use the letter  $\lambda$  for the extension of the regular representation to the group ring  $\mathbb{C}\Gamma$  of  $\Gamma$  i.e.

$$\lambda\left(\sum_{x \in Y} c_x x\right) := \sum_{x \in Y} c_x \lambda(x),$$

for any finite subset  $Y \subset \Gamma$  and any set of complex numbers  $(c_x)_{x \in \Gamma}$  indexed by  $\Gamma$ . The reduced  $C^*$ -algebra of  $\Gamma$  is the  $C^*$ -algebra generated by  $\lambda(\Gamma)$ , i.e.

$$C_r^*(\Gamma) = \overline{\lambda(\mathbb{C}\Gamma)}^{\text{norm}},$$

and the group von Neumann algebra  $L(\Gamma)$  is the von Neumann algebra generated by  $\lambda(\Gamma)$ , i.e.

$$L(\Gamma) = \overline{\lambda(\mathbb{C}\Gamma)}^{\text{so}},$$

where the closure is taken in the strong operator topology on  $B(H)$ . Note that  $L(\Gamma)$  can also be expressed as  $\lambda(\Gamma)''$  (double commutant). Moreover,

$$C_r^*(\Gamma) \subset L(\Gamma) \subset B(\ell^2(\Gamma))$$

with isometric inclusions. We define the norm of an element  $a \in \mathbb{C}\Gamma$  by

$$\|a\| = \|\lambda(a)\|_{B(\ell^2(\Gamma))} = \|\lambda(a)\|_{L(\Gamma)} = \|\lambda(a)\|_{C_r^*(\Gamma)}.$$

Let  $\delta_x(y) := \delta_{x,y}$  for  $x, y \in \Gamma$ . Then  $\{\delta_x \mid x \in \Gamma\}$  form an orthonormal basis for  $\ell^2(\Gamma)$ . The functional

$$(6) \quad \tau(T) = \langle T\delta_e, \delta_e \rangle, \quad T \in L(\Gamma)$$

is a normal faithful tracial state on  $L(\Gamma)$ . In particular,

$$\tau(ST) = \tau(TS), \quad S, T \in L(\Gamma)$$

$$\tau(T^*T) \geq 0, \quad T \in L(\Gamma)$$

$$\tau(T^*T) = 0 \Rightarrow T = 0, \quad T \in L(\Gamma)$$

$$\tau(I) = 1.$$

Since  $\lambda : \mathbb{C}\Gamma \rightarrow C_r^*(\Gamma)$  is one-to-one, we may consider  $\mathbb{C}\Gamma$  as a subalgebra of  $C_r^*(\Gamma) \subset L(\Gamma)$ . Then the trace  $\tau$  restricted to  $\mathbb{C}\Gamma$  satisfies

$$\tau\left(\sum_{x \in Y} c_x x\right) = \begin{cases} c_e, & \text{if } e \in Y \\ 0, & \text{otherwise.} \end{cases}$$

In particular,  $\tau(e) = 1$  and  $\tau(x) = 0$  for  $x \in \Gamma \setminus \{e\}$ . Moreover, for  $a = \sum_{x \in \Gamma} c_x x \in \mathbb{C}\Gamma$ ,

$$(7) \quad \tau(a^*a) = \|a\delta_e\|^2 = \left\| \sum_{x \in \Gamma} c_x \delta_x \right\|^2 = \sum_{x \in \Gamma} |c_x|^2.$$

For all the above see e.g. Section 6.7 of [22] or Section 2.5 of [6]. Let  $S \in L(\Gamma)_{\text{sa}}$  be a self-adjoint operator in  $L(\Gamma)$ . Then

$$\Lambda : C(\sigma(S)) \rightarrow \mathbb{C}$$

given by  $\Lambda(f) = \tau(f(S))$  is a positive functional on  $C(\sigma(S))$  such that  $\Lambda(1) = 1$ . Hence, there is a unique probability measure  $\mu_S$  on  $\sigma(S)$  for which

$$\tau(f(S)) = \int_{\sigma(S)} f d\mu_S, \quad \forall f \in C(\sigma(S)).$$

Since  $\tau$  is a faithful trace,  $\text{supp}(\mu_S) = \sigma(S)$ . We will also consider  $\mu_S$  as a probability measure on all of  $\mathbb{R}$  concentrated on  $\sigma(S)$ . Since  $\|S\| = r(S)$ , the spectral radius of  $S$ , we have

$$(8) \quad \|S\| = \max\{|t| \mid t \in \text{supp}(\mu_S)\} = \max\{|\alpha|, |\beta|\}$$

where  $\alpha = \min(\text{supp}(\mu_S))$  and  $\beta = \max(\text{supp}(\mu_S))$ . By the moments of  $S$  we mean the numbers

$$m_n(S) := \tau(S^n), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Note that  $(m_n(S))_{n=0}^\infty$  are also the moments of the measure  $\mu_S$ , i.e.

$$(9) \quad m_n(S) = \int_{-\infty}^{\infty} t^n d\mu_S(t), \quad n \in \mathbb{N}_0,$$

and since  $\mu_S$  has a compact support,  $\mu_S$  is uniquely determined by its moments. For a not necessarily self-adjoint element  $T \in L(\Gamma)$ , it is convenient to consider the self-adjoint operator

$$\tilde{T} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \in M_2(L(\Gamma)) = L(\Gamma) \otimes M_2(\mathbb{C}),$$

and its moments

$$m_n(\tilde{T}) := \tilde{\tau}(\tilde{T}^n), \quad n \in \mathbb{N}_0,$$

and spectral distribution  $\mu_{\tilde{T}}$  with respect to the normal faithful trace  $\tilde{\tau}$  on  $M_2(L(\Gamma))$  given by  $\tilde{\tau} = \tau \otimes \tau_2$ , where  $\tau_2 = \frac{1}{2}Tr$  on  $M_2(\mathbb{C})$ , i.e.

$$(10) \quad \tilde{\tau} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \frac{\tau(T_{11}) + \tau(T_{22})}{2}, \quad [T_{ij}] \in M_2(L(\Gamma)).$$

Using the trace properties of  $\tau$ , one has

$$(11) \quad m_n(\tilde{T}) = \begin{cases} \tau((T^*T)^{n/2}), & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

Since all the odd moments vanish,  $\mu_{\tau_{\tilde{T}}}$  is symmetric. i.e.  $\check{\mu}_{\tilde{T}} = \mu_{\tilde{T}}$ , where

$$\check{\mu}_{\tilde{T}}(B) = \mu_{\tilde{T}}(-B), \quad B \text{ Borel set in } \mathbb{R}.$$

In particular,  $\min(\text{supp}(\mu_{\tilde{T}})) = -\max(\text{supp}(\mu_{\tilde{T}}))$  and hence by (8)

$$(12) \quad \|T\| = \|\tilde{T}\| = \max(\text{supp}(\mu_{\tilde{T}})),$$

and moreover

$$(13) \quad \pm \|T\| \in \text{supp}(\mu_{\tilde{T}}).$$

Note also that  $\mu_{T^*T}$  is equal to the image measure  $\phi(\mu_{\tilde{T}})$  of  $\mu_{\tilde{T}}$  by the map  $\phi : t \rightarrow t^2$  because  $\mu_{T^*T}$  and  $\phi(\mu_{\tilde{T}})$  are both compactly supported and by (11) they have the same moments.

### 3. COMPUTATIONAL METHODS AND RESULTS

Let  $\Gamma$  be a discrete group and let  $Y \subset \Gamma$  be a finite set with  $|Y| = q + 1$  elements ( $q \geq 2$ ). Our main example will be  $\Gamma = F$ , the Thompson group, and  $Y$  either  $\{I, A, B\}$  or  $\{A, A^{-1}, B, B^{-1}\}$ , but we will for some time stick to the general case in order to treat the two particular cases in a similar way. Our goal is to compute as many as possible of the moments

$$m_n := \tau((h^*h)^n), \quad n \in \mathbb{N}_0,$$

where  $h = \sum_{s \in Y} s \in \mathbb{C}\Gamma$  and  $\tau$  is the trace on  $\mathbb{C}\Gamma$  coming from the group von Neumann algebra  $L(\Gamma)$  as in Section 2. Recall that

$$\tau(x) = \begin{cases} 1, & x = e \\ 0, & x \neq e. \end{cases}$$

Hence  $m_0 = 1$ , and for  $n \geq 1$ , the numbers  $m_n \in \mathbb{N}$  are

$$(14) \quad m_n = |\{(s_1, \dots, s_{2n}) \in Y^{2n} \mid s_1^{-1}s_2 \cdots s_{2n-1}^{-1}s_{2n} = e\}|,$$

where  $|X|$  denotes the number of elements in a set  $X$ . For composing elements of the Thompson group  $F$ , we used the Belk and Brown forest algorithm from [4]. Consider the subsets  $\tilde{E}_k \subset E_k \subset Y^k$

$$(15) \quad E_k := \{(s_1, \dots, s_k) \in Y^k \mid s_1 \neq s_2 \neq \cdots \neq s_k\}$$

$$(16) \quad \tilde{E}_k := \{(s_1, \dots, s_k) \in Y^k \mid s_1 \neq s_2 \neq \cdots \neq s_k \neq s_1\}.$$

We say that  $\tilde{E}_k$  is “cyclic” as all the cyclic rotations of each element in  $\tilde{E}_k$  are contained in the set itself. Since these subsets are smaller, it takes less time to compute the “reduced” numbers

$$(17) \quad \eta_n = |\{(s_1, \dots, s_{2n}) \in E_{2n} \mid s_1^{-1}s_2 \cdots s_{2n-1}^{-1}s_{2n} = e\}|$$

and even less time to compute the “cyclic” numbers

$$(18) \quad \zeta_n = |\{(s_1, \dots, s_{2n}) \in \tilde{E}_{2n} \mid s_1^{-1}s_2 \cdots s_{2n-1}^{-1}s_{2n} = e\}|.$$

The relationship between these numbers and the moments are derived in Section 6. In particular, we have

$$\begin{aligned} \eta_n &= \eta_n - (q-1)(\eta_{n-1} + \eta_{n-2} + \cdots + \eta_1), & n \in \mathbb{N} \\ m_n &= \binom{2n}{n} q^n + \sum_{k=1}^n \binom{2n}{n-k} (\zeta_k + 1 - q) q^{n-k}, & n \in \mathbb{N} \end{aligned}$$

which shows that the first  $n$  moments  $(m_1, \dots, m_n)$  can be computed from either  $(\eta_1, \dots, \eta_n)$  or  $(\zeta_1, \dots, \zeta_n)$ . In the case of a symmetric set  $Y$ , i.e.  $Y = Y^{-1}$ , the reduced numbers  $\eta_n$  are the even co-growth coefficients  $\eta_n = \gamma_{2n}$  introduced by Cohen [11] and Grigorchuk [16]. In the notation of Elder-Rechnitzer-Wong [13],  $m_n = r_{2n}$  and  $\eta_n = \alpha_{2n}$ . (Still for a symmetric set  $Y \subset \Gamma$ ). In Cohen words,  $\eta_n$  is the number of “reduced” words in  $Y$  of length  $2n$  which represent the unit element in  $\Gamma$ .

It is much faster to compute the  $\eta_n$ ’s or  $\zeta_n$ ’s, than to compute the moments  $m_n$  directly from (14). However, we found a different approach to compute the moments, which speeded up the computations much further: Let

$$(19) \quad h_n = \begin{cases} \sum_{(s_1, \dots, s_n) \in E_n} s_1^{-1} s_2 \dots s_{n-1}^{-1} s_n, & (n \text{ even}) \\ \sum_{(s_1, \dots, s_n) \in E_n} s_1 s_2^{-1} \dots s_{n-1}^{-1} s_n, & (n \text{ odd}) \end{cases}$$

and recall that  $\|a\|_2 = \tau(a^*a)^{1/2}$  is the 2-norm on  $L(\Gamma)$  associated with the trace  $\tau$ . Then

$$(20) \quad \tau(h_n^* h_n) = \|h_n\|_2^2,$$

and the reduced number  $\eta_n$  can be computed from the square of the 2-norm of  $h_n$  by the following two equations

$$(21) \quad \begin{aligned} \eta_n &= \xi_n - (q-1)(\xi_{n-1} + \xi_{n-2} + \dots + \xi_1), \\ \xi_n &= \|h_n\|_2^2 - (q+1)q^{n-1} \end{aligned}$$

(cf. Section 6), and hence the moment series  $(m_n)_{n=1}^\infty$  can also be computed from the numbers  $(\|h_n\|_2^2)_{n=1}^\infty$ . In practice we computed  $\|h_n\|_2^2$  as follows. Note first that

$$(22) \quad h_n = \sum_{x \in Y_n} c_x^{(n)} x$$

where  $Y_n \subset \Gamma$  is the set of all distinct terms in the sum (19), and  $c_x^{(n)} \in \mathbb{N}$  is the multiplicity of the occurrence of  $x \in Y_n$  in the sum (19). Note that  $Y_1 = Y$ ,  $Y_2 \subset Y^{-1}Y$ ,  $Y_3 \subset YY^{-1}Y$ ,  $Y_4 \subset Y^{-1}YY^{-1}Y$ , etc. Since

$$\tau(x^{-1}y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

we have

$$(23) \quad \|h_n\|_2^2 = \sum_{x \in Y_n} (c_x^{(n)})^2.$$

The advantage of computing the squared 2-norms  $\|h_n\|_2^2$  instead of the reduced numbers  $\eta_n$  or the cyclic numbers  $\zeta_n$  is that we only have to consider  $(1+q)q^{n-1}$  words of length  $n$  instead of  $(1+q)q^{2n-1}$  words of length  $2n$ . This made it possible for us to almost double the number of moments we could compute in the two cases  $Y = \{I, A, B\}$  and  $Y = \{A, B, A^{-1}, B^{-1}\}$  for the Thompson group  $F$ . The only drawback was that we need first to store all the terms of the sum (19), and next to sort the list in order to compute the multiplicities  $c_x^{(n)} \in \mathbb{N}$ ,  $x \in Y_n$ .

We wrote two programs, both using parallel computing, to calculate the squared 2-norms  $\|h_n\|_2^2$ , which can be downloaded at:



<https://github.com/shaagerup/ThompsonGroupF/>  
<https://github.com/mariars/ThompsonGroupF/>  
<http://www.math.ku.dk/~haagerup/ThompsonGroupF/>  
<http://www.math.ku.dk/~mrs/ThompsonGroupF/>

The first program was written in Haskell, where the code loops through all the terms in the sum (19) and saves them to the hard disk. We then use the GNU-sort program to find the multiplicities. This program was run in a supercomputer with 32 cores and 128 GB memory at the University of Copenhagen. The second program was written in  $C\#$ . For the case  $Y = \{A, B, A^{-1}, B^{-1}\}$ , we made one further step to reduce both the computation time and the size of the storing data. Since  $|E_n| = 4 \cdot 3^{n-1}$  in this case, the computation time will increase at least by a factor of 3 when going from  $n$  to  $n+1$ . In practice the computing time increased by

TABLE 1. The series of numbers for  $h = I + A + B$  (Case 1).

$n$	$\ h_n\ _2^2$	$\eta_n$	$\zeta_n$	$m_n(h^*h)$
1	3	0	0	3
2	6	0	0	15
3	12	0	0	87
4	24	0	0	543
5	48	0	0	3543
6	96	0	0	23823
7	192	0	0	163719
8	400	16	16	1144015
9	800	16	0	8100087
10	1656	72	40	57971735
11	3344	104	0	418640071
12	7032	448	240	3046373007
13	14272	656	0	22314896087
14	30544	2656	1344	164407579407
15	63120	4688	720	1217526417687
16	137264	15712	7056	9057960864015
17	292160	33344	8976	67667981453831
18	651960	100984	43272	507425879338551
19	1435808	232872	74176	3818200408513415
20	3310592	671848	280280	28821799875573303
21	7593024	1643688	580272	218200189786794855
22	18161528	4619168	1912064	1656415132760705871
23	43488112	11784224	4457952	12606151256856370471
24	107764880	32572880	13462384	96166410605134544815
25	268721056	85764176	34080800	735237884585469467543
26	686850128	235172192	97724640	5632983879577272289359
27	1769246208	630718144	258098400	43241777428163458121799
28	4640551024	1732776752	729438864	332564656181337723832623
29	12254456800	4706131504	1970016864	2562203165206920141303479
30	32773003720	12970221624	5527975480	1977320516001075298777543
31	88160278544	35584492728	15172024960	152837887007013006956440295
32	239251904104	98515839744	42518879248	1183157961642417140248556303
33	652453973392	272466004928	117953204688	9172380845538923831902240519
34	1790526123576	758084181720	331105376552	71206765648586031626111809367
35	4933923852880	2110955787448	925892800560	553521536480845628126004101879
36	13660080583776	5903188665464	2607169891128	4308220957036953495382444267287
37	37952694315360	16535721813272	7336514373472	33572939291063083015187615095255

TABLE 2. The series of numbers for  $h = A + B + A^{-1} + B^{-1}$  (Case 2).

$n$	$\ h_n\ _2^2$	$\eta_n$	$\zeta_n$	$m_n(h^*h)$
1	4	0	0	4
2	12	0	0	28
3	36	0	0	232
4	108	0	0	2092
5	344	20	20	19884
6	1076	64	24	196096
7	3500	336	168	1988452
8	11324	1160	320	20612364
9	38708	5896	2736	217561120
10	134880	24652	9700	2331456068
11	497616	117628	53372	25311956784
12	1906356	531136	231624	277937245744
13	7747484	2559552	1197768	3082543843552
14	32825220	12142320	5661432	34493827011868
15	145750148	59416808	28651280	389093033592912
16	668749196	290915560	141316416	4420986174041164
17	3164933480	1449601452	718171188	50566377945667804
18	15314270964	7269071976	3638438808	581894842848487960
19	75551504916	36877764000	18708986880	6733830314028209908
20	378261586048	188484835300	96560530180	78331435477025276852
21	1918303887820	972003964976	503109989256	915607264080561034564
22	9831967554120	5049059855636	2636157949964	10750847942401254987096
23	50870130025336	26423287218612	13912265601668	126768974481834814357308
24	265393048436340	139205945578944	73848349524776	1500753741925909645997904

a factor  $\approx 3.2$ . It is known (cf. [12], [17]) that the size of the spheres  $S_n$  of radius  $n$  in the word metric of  $F$ , grows roughly with a factor  $\phi^2 = 2.618\dots$  (the square of the golden ratio  $\phi = \frac{1+\sqrt{5}}{2}$ ), when passing from  $S_n$  to  $S_{n+1}$ . Hence we expect the size of  $Y_n \subset Y^n = (S_1)^n = S_n \sqcup S_{n-2} \sqcup S_{n-4} \sqcup \dots$  to grow with a factor close to 2.618 when passing from  $n$  to  $n+1$ . By using the recursion formula

$$h_{n+1} = h \cdot h_n - qh_{n-1}$$

derived in Section 6 (see formulas (51)-(52) for the case  $h_n = k_n$ ) we could compute the list of multiplicities  $(c_x^{n+1})_{x \in Y_{n+1}}$  from the lists  $(c_x^n)_{x \in Y_n}$  and  $(c_x^{n-1})_{x \in Y_{n-1}}$ , and by this method the computation time only grew  $\approx 2.8$  when passing from  $n$  to  $n+1$ . This worked well for computing the numbers  $\|h_n\|_2^2$  for  $n \leq 23$ . To obtain  $\|h_{24}\|_2^2$  an extra trick was implemented to speed up the computations and to reduce the size of the storing data. The elements of  $Y_n \subset \Gamma$  are homeomorphisms written as forest diagrams  $(u, v)$  which consist of two “reduced” trees – the domain tree  $u$ , and the range tree  $v$ . The extra trick consists on the observation that if  $(u, v) \in Y_n \setminus \{e\}$  then so is its inverse  $(v, u) \in Y_n \setminus \{e\}$ . Saving only one of them reduces the storing size by about one half. This program was run in a standard desktop computer with an AMD Phenom II X4 965 Quad-core processor (3400 Mhz) and 2 TB of SSD-hard disk. It took about 5 days to compute and store the forest diagrams (in serialized form) for  $\|h_{24}\|_2^2$  for  $Y = \{A, B, A^{-1}, B^{-1}\}$ , and another five days to do the sorting. The series of numbers  $\|h_n\|_2^2$ ,  $\eta_n$ ,  $\zeta_n$ ,  $m_n(h^*h)$  for case 1 (resp. case 2) are shown in Table 1 (resp. Table 2).

In Appendix B (Theorem B.1(iii)) we show that if  $\Gamma$  is torsion free (e.g. if  $\Gamma = F$ ), then the numbers

$$\zeta'_n := \sum_{d|n} \mu\left(\frac{n}{d}\right) \zeta_d, \quad n \in \mathbb{N}$$

are non negative integers divisible by  $2n$ . Here  $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$  is the Möbius function (cf. Section 2.2. in [2]). We used this as a test to detect possible programming errors in the computation of the sequence of numbers  $\|h_n\|_2^2$ . This test rules out false values of  $\|h_n\|_2^2$  with probability  $1 - \frac{1}{2n}$ .

#### 4. ESTIMATING NORMS

Let  $T \in L(\Gamma)$  be an operator in the von Neumann algebra of a countable group  $\Gamma$ . We will in this section discuss how we can get good lower bounds for  $\|T\|$  from knowing only finitely many of the moments  $m_n(T^*T) = \tau((T^*T)^n)$  as well as getting some prediction of what the actual values of  $\|T\|$  might be. The following proposition is well known. For the convenience of the reader, we include a proof.

**Proposition 4.1.** *Let  $T \in L(\Gamma)$ ,  $T \neq 0$ . Then*

$$(24) \quad m_n(T^*T)^{1/n} \leq \frac{m_n(T^*T)}{m_{n-1}(T^*T)} \leq \|T\|^2 \quad n \in \mathbb{N}.$$

Moreover, both  $(m_n^{1/n})_{n=1}^\infty$  and  $(\frac{m_n}{m_{n-1}})_{n=1}^\infty$  are increasing sequences and

$$(25) \quad \lim_{n \rightarrow \infty} m_n(T^*T)^{1/n} = \lim_{n \rightarrow \infty} \frac{m_n(T^*T)}{m_{n-1}(T^*T)} = \|T\|^2.$$

*Proof.* Let  $\nu = \mu_{T^*T}$  be the spectral distribution measure of  $T^*T$  with respect to the trace  $\tau$  on  $L(\Gamma)$ . Since  $T^*T \geq 0$  is a positive self-adjoint operator, we have from Section 2 that

$$\text{supp}(\nu) = \sigma(T^*T) \subset [0, \infty)$$

and

$$\max(\text{supp}(\nu)) = \max\{|x| \mid x \in \text{supp}(\nu)\} = \|T^*T\| = \|T\|^2.$$

Let  $m_n := m_n(T^*T)$ ,  $n \in \mathbb{N}_0$ . Then

$$m_n = \int_0^{\|T\|^2} t^n d\nu(t) > 0$$

because  $\text{supp}(\nu) \neq \{0\}$  as  $T \neq 0$ . Using  $t^n \leq \|T\|^2 t^{n-1}$  for  $t \in [0, \|T\|^2]$ , we get  $m_n/m_{n-1} \leq \|T\|^2$  for  $n \geq 1$ . Moreover by Hölder's inequality, we have

$$\begin{aligned} m_n &= \int_0^{\|T\|^2} (t^{n-1})^{1/2} (t^{n+1})^{1/2} d\nu(t) \\ &\leq \left( \int_0^{\|T\|^2} t^{n-1} d\nu(t) \right)^{1/2} \left( \int_0^{\|T\|^2} t^{n+1} d\nu(t) \right)^{1/2} \\ &\leq m_{n-1}^{1/2} \cdot m_{n+1}^{1/2}. \end{aligned}$$

Hence,  $m_n/m_{n-1} \leq m_{n+1}/m_n$ , i.e.  $(m_n/m_{n-1})_{n=1}^\infty$  is an increasing sequence. Since  $m_0 = 1$ ,

$$m_n = \frac{m_1}{m_0} \frac{m_2}{m_1} \cdots \frac{m_n}{m_{n-1}} \leq \left( \frac{m_n}{m_{n-1}} \right)^n$$

which proves (24). Using Hölder's inequality we have

$$m_n = \int_0^{\|T\|^2} t^n \cdot 1 \, d\nu(t) \leq \left( \int_0^{\|T\|^2} t^{n+1} \, d\nu(t) \right)^{n/(n+1)} \left( \int_0^{\|T\|^2} 1 \, d\nu(t) \right)^{1/(n+1)} = m_{n+1}^{n/(n+1)}$$

which proves that  $(m_n^{1/n})_{n=1}^\infty$  is an increasing sequence. Let  $0 < a < \|T\|^2$ . Since  $\|T\|^2 \in \text{supp}(\nu)$ ,

$$\nu([a, \|T\|^2]) > 0.$$

Since

$$m_n \geq \int_a^{\|T\|^2} t^n \, d\nu(t) \geq a^n \nu([0, \|T\|^2]).$$

It follows that  $\liminf_{n \rightarrow \infty} (m_n^{1/n}) \geq a$  for all  $a \in (0, \|T\|^2)$ . Hence  $\liminf_{n \rightarrow \infty} (m_n^{1/n}) \geq \|T\|^2$  which together with (24) proves (25).  $\square$

The lower bounds  $m_n (T^*T)^{\frac{1}{2n}}$  and  $\frac{m_n (T^*T)^{1/2}}{m_{n-1} (T^*T)^{1/2}}$  for  $\|T\|$  in Proposition 4.1 give, however, rather poor lower estimates of  $\|T\|$  in the two main cases we consider, namely

$$T_1 = I + A + B, \quad T_2 = A + A^{-1} + B + B^{-1}$$

in the group ring  $\mathbb{C}F$  of the Thompson group  $F$ . To get better estimates, we apply methods from the theory of orthogonal polynomials (c.f [33] or [20]) to the symmetric measure  $\mu_{\tilde{T}}$  described in Section 2.

Let  $\mu$  be a compactly supported Radon measure on  $\mathbb{R}$  for which  $\text{supp}(\mu)$  is not a finite set. Then the polynomials  $1, t, t^2, \dots$  are linear independent in  $L^2(\mu)$ . Hence by Gram-Schmidt orthonormalization we get a sequence of polynomials  $(p_n)_{n=0}^\infty$  of degree  $(p_n) = n$ , such that

$$\int_{\mathbb{R}} p_m(t) p_n(t) \, d\mu(t) = \delta_{m,n}, \quad m, n \in \mathbb{N}_0.$$

Moreover, the polynomials  $(p_n)_{n=0}^\infty$  are uniquely determined if we add the condition that  $k_n > 0$ , where  $k_n$  is the coefficient of  $t^n$  in the polynomial  $p_n(t)$ . Let

$$c_n := \int_{\mathbb{R}} t^n \, d\mu(t), \quad n \in \mathbb{N}_0$$

be the  $n$ 'th moment of  $\mu$ , and put

$$(26) \quad D_n := \det([c_{i+j}]_{i,j=0}^n).$$

Then by Formula (2.2.7) and (2.2.15) in [33] we have

$$(27) \quad D_n > 0 \quad n \geq 0$$

$$(28) \quad k_n = \left( \frac{D_{n-1}}{D_n} \right)^{1/2} \quad n \geq 1.$$

Note that  $k_0 = c_0^{-1/2} = D_0^{-1/2}$ . Hence (28) holds for all  $n \geq 0$  if we define

$$(29) \quad D_{-1} := 1.$$

Note also, that since  $\mu$  is compactly supported, the set of polynomials is dense in  $L^2(\mu)$ . Hence  $(p_n)_{n=0}^\infty$  is an orthonormal basis for the Hilbert space  $L^2(\mu)$ .

If we furthermore assume that  $\mu$  is a symmetric measure on  $\mathbb{R}$  (i.e.  $\mu(-B) = \mu(B)$  for every Borel set  $B$ ), then it is easily seen that

$$(30) \quad p_n(-t) = (-1)^n p_n(t), \quad n \in \mathbb{N}_0.$$

An elementary application of the recursion formula for orthonormal polynomials (c.f Section 3.2 in [33]) now gives:

**Proposition 4.2.** *Let  $\mu$  be a symmetric compactly supported Radon measure on  $\mathbb{R}$  for which the support is not finite. Then the multiplication operator*

$$m_t : f(t) \rightarrow tf(t), \quad f \in L^2(\mu)$$

*in  $B(L^2(\mu))$  has the form*

$$(31) \quad M = [m_{ij}]_{i,j=0}^{\infty} = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & \cdots \\ \alpha_1 & 0 & \alpha_2 & 0 & \cdots \\ 0 & \alpha_2 & 0 & \alpha_3 & \cdots \\ 0 & 0 & \alpha_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

*with respect to the orthonormal basis  $(p_n)_{n=0}^{\infty}$  for  $L^2(\mu)$ , where*

$$(32) \quad \alpha_n = \frac{k_{n-1}}{k_n} = \left( \frac{D_{n-2}D_n}{D_{n-1}^2} \right), \quad n \geq 1.$$

*Proof.* From Theorem 3.2.1 in [33] or pp. 58-59 in [20] we have

$$p_{n+1}(t) = (A_n t + B_n)p_n(t) - C_n p_{n-1}(t), \quad n \geq 1$$

and

$$p_1(t) = (A_0 t + B_0)p_0(t),$$

where  $A_n = \frac{k_{n+1}}{k_n}$  ( $n \geq 0$ ) and  $C_n = \frac{A_n}{A_{n-1}}$  ( $n \geq 1$ ). Moreover,  $B_n = 0$  for all  $n \geq 0$  by (30). Hence

$$\begin{aligned} tp_n(t) &= \frac{1}{A_n} p_{n+1}(t) + \frac{C_n}{A_n} p_{n-1}(t) \\ &= \frac{k_n}{k_{n+1}} p_{n+1}(t) + \frac{k_{n-1}}{k_n} p_{n-1}(t), \quad n \geq 1 \end{aligned}$$

and

$$tp_0(t) = \frac{k_0}{k_1} p_1(t).$$

This shows that the matrix for  $m_t$  with respect to the basis  $(p_n)_{n=0}^{\infty}$  has the form (31) where  $\alpha_n = \frac{k_{n-1}}{k_n}$  ( $n \geq 1$ ). The second formula for  $\alpha_n$  in (32) follows from (28) and (29).  $\square$

**Proposition 4.3.** *Let  $\mu$  and  $M$  be as in Proposition 4.2. Let*

$$M_n = [m_{ij}]_{i,j=0}^n = \begin{pmatrix} 0 & \alpha_1 & & & \\ \alpha_1 & 0 & \alpha_2 & & 0 \\ & \alpha_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \alpha_n \\ 0 & & & \alpha_n & 0 \end{pmatrix}$$

*and let  $\|M\|$  and  $\|M_n\|$  denote the norm of  $M$  (resp.  $M_n$ ) considered as operators on  $\ell^2(\mathbb{N}_0)$  (resp.  $\ell^2(\{0, \dots, n\})$ ). Then*

$$(33) \quad \|M\| = \max(\text{supp}(\mu))$$

$$(34) \quad \|M_n\| = \lambda_{\max}(M_n),$$

where  $\lambda_{\max}$  denotes the largest eigenvalue of the symmetric matrix  $M_n$ . Moreover,

$$(35) \quad (||M_n||)_{n=1}^{\infty} \text{ is an increasing sequence.}$$

$$(36) \quad \lim_{n \rightarrow \infty} ||M_n|| = ||M||.$$

*Proof.* Since  $(p_n)_{n=0}^{\infty}$  is an orthonormal basis for  $L^2(\mu)$

$$m_t = U M U^*$$

where  $U$  is the unitary matrix from  $\ell^2(\mathbb{N}_0)$  to  $L^2(\mu)$  for which  $U\delta_n = p_n$ , where  $(\delta_n)_{n=0}^{\infty}$  is the standard basis for  $\ell^2(\mathbb{N}_0)$ . Hence

$$||M|| = ||m_t||_{B(L^2(\mu))}.$$

But the norm of the multiplication operator  $m_t$  is just the  $L^\infty(\mu)$ -norm of the function  $t \rightarrow t$  ( $t \in \mathbb{R}$ ). Hence

$$||M|| = \max\{|x| \mid x \in \text{supp}(\mu)\} = \max(\text{supp}(\mu)),$$

where the last equality follows from the assumption that  $\mu$  is symmetric. This proves (33). Since  $M_n$  is a self-adjoint matrix

$$||M_n|| = \max\{|\lambda| \mid \lambda \text{ eigenvalue of } M_n\}.$$

Since  $G_n M_n G_n^{-1} = -M_n$ , where  $G_n$  is the diagonal matrix:

$$G_n = \text{diag}(1, -1, 1, -1, \dots, (-1)^n),$$

the set of eigenvalues of  $M_n$  and  $-M_n$  are the same. Hence

$$||M_n|| = \lambda_{\max}(M_n) = -\lambda_{\min}(M_n)$$

proving (34). Finally (35) and (36) follows from the fact that  $\ell^2\{(0, \dots, n)\}$  is an increasing sequence of subspaces of  $\ell^2(\mathbb{N}_0)$  whose union is dense in  $\ell^2(\mathbb{N}_0)$ .  $\square$

**Proposition 4.4.** *Let  $\mu$  and  $M$  be as in Proposition 4.2. Then*

$$(37) \quad \liminf_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n) \leq ||M|| \leq \sup_{n \geq 2} (\alpha_{n-1} + \alpha_n).$$

If  $\frac{1}{\sqrt{2}}\alpha_1 \leq \alpha_2 \leq \alpha_3$  then

$$(38) \quad ||M|| \leq \sup_{n \geq 3} (\alpha_{n-1} + \alpha_n).$$

*Proof.* Let

$$a = \liminf_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n) = \liminf_{n \rightarrow \infty} (\alpha_n + \alpha_{n+1}).$$

Let  $\varepsilon > 0$  and choose  $n_0 \in \mathbb{N}$  such that  $\alpha_n + \alpha_{n+1} \geq a - \varepsilon$  for  $n \geq n_0$ . For  $k \in \mathbb{N}$  consider the unit vector  $x = (x_i)_{i=0}^{\infty} \in \ell^2(\mathbb{N}_0)$  given by

$$x_i = \begin{cases} (2k+1)^{-1/2}, & n_0 \leq i \leq n_0 + 2k \\ 0, & \text{otherwise.} \end{cases}$$

Let  $\langle \cdot, \cdot \rangle$  denote the inner product in  $\ell^2(\mathbb{N}_0)$ . Then

$$||M|| \geq \langle Mx, x \rangle = \frac{2}{2k+1} (\alpha_{n_0} + \alpha_{n_0+1} + \dots + \alpha_{n_0+2k-1}) \geq \frac{2k}{2k+1} (a - \varepsilon).$$

Letting first  $k \rightarrow \infty$  and next  $\varepsilon \rightarrow 0$  we get  $||M|| \geq a$  proving the first inequality in (37). Since  $M = [m_{ij}]_{i,j=0}^{\infty}$  is a symmetric matrix with non-negative entries,

it follows from Schur's test (see e.g. Exercise 3.2.17 in [29]) that if there exists a sequence  $(q_n)_{n=0}^\infty$  of strictly positive real numbers and a constant  $C > 0$  such that

$$\sum_{j=0}^{\infty} m_{ij} q_j \leq C q_i, \quad i \in \mathbb{N}_0$$

then  $\|M\| \leq C$ . Applying this to the constant sequence  $q_n = 1$ ,  $n \in \mathbb{N}_0$ , we get

$$\|M\| \leq \sup\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \dots\}$$

proving the second inequality in (37). If we instead let  $q_0 = \frac{1}{\sqrt{2}}$  and  $q_n = 1$  for all  $n \geq 1$ , we get

$$\|M\| \leq \sup\{\sqrt{2}\alpha_1, \frac{1}{\sqrt{2}}\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \dots\}.$$

Under the assumptions of (38),

$$\max\{\sqrt{2}\alpha_1, \frac{1}{\sqrt{2}}\alpha_1 + \alpha_2\} \leq 2\alpha_2 \leq \alpha_2 + \alpha_3.$$

Hence (38) holds.  $\square$

**Remark 4.5.** Note that for  $T \in L(\Gamma)$  we have from Section 2, that

$$m_n(T^*T) = \int_{-||T||}^{||T||} t^{2n} d\mu_{\tilde{T}}(t) dt, \quad n \in \mathbb{N}_0,$$

and  $\mu_{\tilde{T}}$  is a symmetric compactly supported probability measure on  $\mathbb{R}$  and

$$||T|| = \max(\text{supp}(\mu_{\tilde{T}})).$$

Hence, when applying Propositions 4.2, 4.3 and 4.4 to  $\mu = \mu_{\tilde{T}}$  we have

$$(39) \quad c_{2n} = m_n(T^*T) \quad \text{and} \quad c_{2n+1} = 0, \quad n \in \mathbb{N}_0$$

$$(40) \quad ||T|| = \|M\| = \lim_{n \rightarrow \infty} \|M_n\|$$

and if  $\frac{1}{\sqrt{2}}\alpha_1 \leq \alpha_2 \leq \alpha_3$ ,

$$(41) \quad \liminf_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n) \leq ||T|| \leq \sup_{n \geq 3} (\alpha_{n-1} + \alpha_n).$$

The condition  $\frac{1}{\sqrt{2}}\alpha_1 \leq \alpha_2 \leq \alpha_3$  is fulfilled in the two main cases that we consider, namely  $T_1 = I + A + B$ , and  $T_2 = A + A^{-1} + B + B^{-1}$  (see Tables 3-4). Actually in these two cases, the right hand side of (41) can be sharpened further to

$$(42) \quad ||T|| \leq \limsup_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n)$$

(See Appendix A, Corollary A.8).

**Remark 4.6.** If  $\mu$  is finitely supported, then  $L^2(\mathbb{R}, \mu)$  is finite dimensional and

$$d = \dim L^2(\mathbb{R}, \mu) = |\text{supp}(\mu)|.$$

Therefore, the Gram-Schmidt orthonormalization process will end after  $d - 1$  steps and we get a finite matrix

$$M = \begin{pmatrix} 0 & \alpha_1 & & & \\ \alpha_1 & 0 & \alpha_2 & & 0 \\ & \alpha_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \alpha_{d-1} \\ 0 & & & \alpha_{d-1} & 0 \end{pmatrix}$$

where the numbers  $\alpha_n$  ( $1 \leq n \leq d - 1$ ) are given by (32). This will however not happen in the two main cases we are interested in, (see Appendix A, Corollary A.4).

The following proposition extends Proposition 4.1.

**Proposition 4.7.** *Let  $T \in L(\Gamma)$ ,  $T \neq 0$ , and let  $(M_n)_{n=1}^\infty$  be as in Remark 4.5. Then*

$$\frac{m_n(T^*T)}{m_{n-1}(T^*T)} \leq \|M_n\|^2 = (\lambda_{\max}(M_n))^2 \leq \|T\|^2, \quad n \in \mathbb{N}.$$

Let  $\delta_0, \delta_1, \delta_2, \dots$  denote the standard orthonormal basis for  $\ell^2(\mathbb{N}_0)$  (i.e.  $\delta_0 = (1, 0, 0, \dots)$ ), and  $\delta'_0, \delta'_1, \delta'_2, \dots, \delta'_n$  denote the standard basis for  $\ell^2(\{0, 1, \dots, n\})$ . For the proof of Proposition 4.7 we will need the following lemma:

**Lemma 4.8.** *For  $n \in \mathbb{N}_0$  and  $k = 0, \dots, n$ , the moments  $m_k(T^*T)$  are given by*

$$m_k(T^*T) = \|(M_n)^k \delta'_0\|_2^2.$$

*Proof.* Recall that  $p_0 = 1$  as it is the first element of the orthonormal basis (i.e. first step in Gram-Schmidt). Since  $p_0, \dots, p_n$  is an orthonormalization of  $1, t, \dots, t^n$ , we have

$$(m_t)^n p_0 = (m_t)^n 1 = t^n \in \text{span}(1, t, \dots, t^n) = \text{span}(p_0, p_1, \dots, p_n).$$

Thus for  $k = 0, \dots, n$  we have

$$M^k \delta_0 \in \text{span}(\delta_0, \dots, \delta_n)$$

and

$$\|M^k \delta_0\|_2^2 = \|(m_t)^k p_0\|_2^2 = \|t^k\|_2^2 = \int_{-||T||}^{||T||} t^{2k} d\mu_{\tilde{T}}(t) = m_k(T^*T),$$

as  $M$  is the matrix of  $m_t$  with respect to the ONB  $p_0, p_1, \dots$ . Let  $E_n$  be the projection of  $\ell^2(\mathbb{N}_0)$  onto  $\text{span}(\delta_0, \dots, \delta_n)$ . We claim that

$$(E_n M E_n)^k \delta_0 = M^k \delta_0.$$

Since

$$E_n M E_n = \begin{pmatrix} M_n & 0 \\ 0 & 0 \end{pmatrix},$$

it follows from the claim that

$$\|(M_n)^k \delta'_0\|_2^2 = \|(E_n M E_n)^k \delta_0\|_2^2 = \|M^k \delta_0\|_2^2 = m_k(T^*T),$$

which proves the Lemma. The proof of the claim is done by induction in  $k$ . For  $k = 1$  we have  $E_n M E_n \delta_0 = E_n M \delta_0 = M \delta_0$  because  $M \delta_0 \in \text{span}\{\delta_0, \delta_1\}$  and  $n \geq 1$ . Assume next that  $(E_n M E_n)^k \delta_0 = M^k \delta_0$  for  $k \leq n - 1$ . Then  $(E_n M E_n)^{k+1} \delta_0 = E_n M E_n M^k \delta_0 = E_n M^{k+1} \delta_0 = M^{k+1} \delta_0$  as  $M^k \delta_0 \in \text{span}(\delta_0, \dots, \delta_n)$ , which proves the claim.  $\square$



*Proof of Proposition 4.7.* By the previous Lemma 4.8 we have

$$m_n = \|(M_n)^n \delta'_0\|_2^2 = \|M_n(M_n)^{n-1} \delta'_0\|_2^2 \leq \|M_n\|^2 \|(M_n)^{n-1} \delta'_0\|_2^2 = \|M_n\|^2 m_{n-1}$$

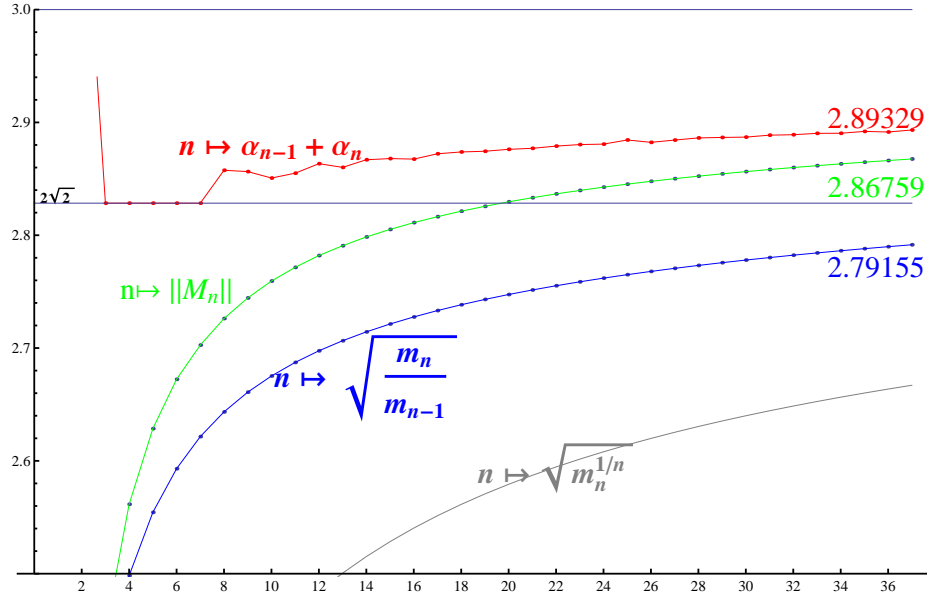
Hence the first inequality of the proposition. The rest of the proof of Proposition 4.7 follows from Proposition 4.3 and Remark 4.5.  $\square$

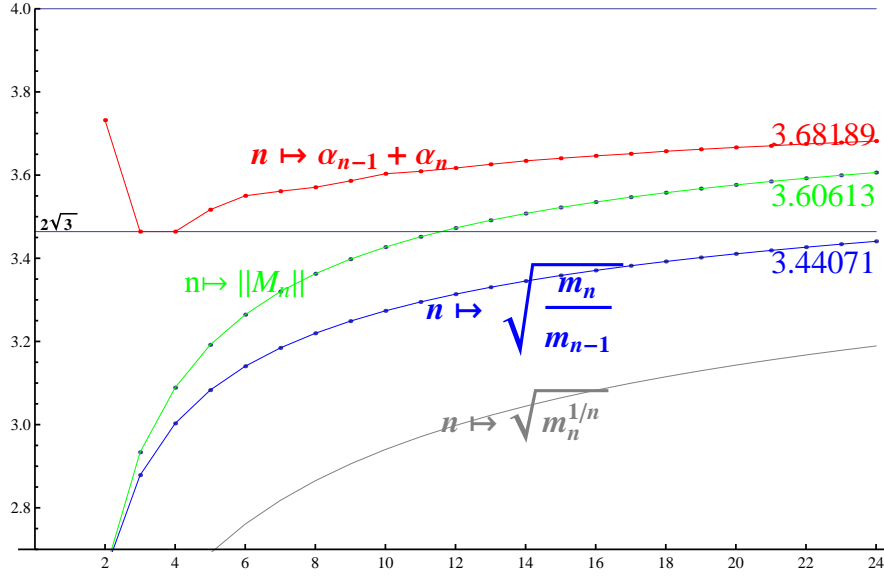
TABLE 3. Estimating the norm  $\|I + A + B\|$ .

$n$	$m_n^{\frac{1}{2^n}}$	$\left(\frac{m_n}{m_{n-1}}\right)^{\frac{1}{2}}$	$\alpha_n$	$\lambda_{\max}(M_n)$	$\alpha_{n-1} + \alpha_n$
1	1.73205	1.73205	1.73205	1.73205	— — —
2	1.96799	2.23607	1.41421	2.23607	3.14626
3	2.10501	2.40832	1.41421	2.44949	2.82843
4	2.19710	2.49828	1.41421	2.56155	2.82843
5	2.26432	2.55438	1.41421	2.62860	2.82843
6	2.31606	2.59306	1.41421	2.67233	2.82843
7	2.35741	2.62151	1.41421	2.70265	2.82843
8	2.39140	2.64342	1.44338	2.72620	2.85759
9	2.41994	2.66090	1.41303	2.74445	2.85641
10	2.44434	2.67524	1.43768	2.75941	2.85071
11	2.46548	2.68728	1.41733	2.77154	2.85500
12	2.48403	2.69756	1.44610	2.78200	2.86343
13	2.50048	2.70649	1.41406	2.79072	2.86016
14	2.51518	2.71434	1.45286	2.79850	2.86691
15	2.52842	2.72131	1.41509	2.80515	2.86795
16	2.54043	2.72757	1.45239	2.81118	2.86749
17	2.55138	2.73323	1.41982	2.81645	2.87222
18	2.56143	2.73839	1.45400	2.82132	2.87382
19	2.57069	2.74311	1.42044	2.82563	2.87444
20	2.57925	2.74746	1.45571	2.82966	2.87615
21	2.58720	2.75148	1.42133	2.83326	2.87704
22	2.59461	2.75522	1.45768	2.83665	2.87901
23	2.60154	2.75871	1.42269	2.83972	2.88037
24	2.60803	2.76198	1.45807	2.84263	2.88076
25	2.61414	2.76505	1.42638	2.84529	2.88445
26	2.61989	2.76793	1.45596	2.84782	2.88234
27	2.62533	2.77066	1.42841	2.85014	2.88437
28	2.63047	2.77323	1.45785	2.85237	2.88626
29	2.63535	2.77568	1.42883	2.85443	2.88669
30	2.63998	2.77800	1.45815	2.85641	2.88698
31	2.64439	2.78021	1.43056	2.85824	2.88871
32	2.64860	2.78231	1.45854	2.86002	2.88910
33	2.65261	2.78432	1.43178	2.86167	2.89032
34	2.65645	2.78625	1.45860	2.86327	2.89039
35	2.66012	2.78809	1.43344	2.86477	2.89204
36	2.66365	2.78986	1.45806	2.86623	2.89150
37	2.66702	2.79155	1.43523	2.86759	2.89329

TABLE 4. Estimating the norm  $\|A + B + A^{-1} + B^{-1}\|$ .

$n$	$m_n^{\frac{1}{2n}}$	$\left(\frac{m_n}{m_{n-1}}\right)^{\frac{1}{2}}$	$\alpha_n$	$\lambda_{\max}(M_n)$	$\alpha_{n-1} + \alpha_n$
1	2.00000	2.00000	2.00000	2.00000	— — —
2	2.30033	2.64575	1.73205	2.64575	3.73205
3	2.47884	2.87849	1.73205	2.93352	3.46410
4	2.60058	3.00287	1.73205	3.08891	3.46410
5	2.69061	3.08298	1.78471	3.19184	3.51676
6	2.76083	3.14038	1.76554	3.26439	3.55025
7	2.81769	3.18437	1.79564	3.32000	3.56119
8	2.86506	3.21963	1.77500	3.36276	3.57064
9	2.90535	3.24883	1.81110	3.39790	3.58610
10	2.94023	3.27358	1.79214	3.42682	3.60324
11	2.97083	3.29495	1.81693	3.45164	3.60907
12	2.99800	3.31368	1.80006	3.47272	3.61699
13	3.02234	3.33028	1.82585	3.49133	3.62591
14	3.04432	3.34515	1.80841	3.50755	3.63425
15	3.06433	3.35858	1.83203	3.52217	3.64043
16	3.08264	3.37080	1.81426	3.53513	3.64629
17	3.09949	3.38198	1.83702	3.54697	3.65129
18	3.11507	3.39228	1.82036	3.55761	3.65739
19	3.12954	3.40180	1.84173	3.56745	3.66210
20	3.14303	3.41065	1.82478	3.57639	3.66651
21	3.15565	3.41890	1.84556	3.58471	3.67034
22	3.16749	3.42663	1.82942	3.59235	3.67498
23	3.17863	3.43388	1.84878	3.59952	3.67820
24	3.18914	3.44071	1.83311	3.60613	3.68189

FIGURE 2. Estimating the norm  $\|I + A + B\|$ .

FIGURE 3. Estimating the norm  $\|A + B + A^{-1} + B^{-1}\|$ 

Recall that from Proposition 4.1 and Proposition 4.7 that

$$m_n^{\frac{1}{2n}} \leq \left( \frac{m_n}{m_{n-1}} \right)^{1/2} \leq \lambda_{\max}(M_n) \leq \|T\|,$$

and all three sequences on the left hand side converges monotonically to  $\|T\|$  as  $n \rightarrow \infty$ . Note that in both cases  $m_n^{\frac{1}{2n}}$  and  $(m_n/m_{n-1})^{1/2}$  are poor lower estimates (see Tables 3, 4 and Figs. 2, 3). For the listed range of integers  $n$ , they both stay well below the known lower bound  $\|I + A + B\| > 2\sqrt{2} \approx 2.82824$  in case 1, (resp.  $\|A + A^{-1} + B + B^{-1}\| > 2\sqrt{3} \approx 3.46410$  in case 2), while the lower estimates  $\lambda_{\max}(M_n)$  stay above this value for  $n \geq 20$  in case 1 (resp. for  $n \geq 12$  in case 2). The best exact lower bound for  $\|T\|$  in case 1 ( $T = I + A + B$ ) we can obtain from our results is

$$\|I + A + B\| \geq \lambda_{\max}(M_{37}) = 2.86759.$$

Note however that by Proposition 4.4,

$$\|I + A + B\| \geq \liminf_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n)$$

and since  $(\alpha_{n-1} + \alpha_n)_{n=1}^{\infty}$  appear to be monotonically increasing for  $n \geq 26$  our computation results make it very likely that actually

$$\|I + A + B\| \geq \alpha_{36} + \alpha_{37} = 2.89329.$$

To get some prediction of the actual value of  $\|I + A + B\|$ , we made a least squares fitting of the 26 numbers

$$\lambda_{\max}(M_n), \quad 12 \leq n \leq 37$$

to a function of the following form

$$f(n) = a - b(n - c)^{-d}, \quad n = 12, \dots, 37$$

and found that the optimal values of the parameters  $a, b, c, d$  were

$$a = 2.950 \quad b = 0.630 \quad c = 0.571 \quad d = 1.900.$$

In particular, this extrapolation argument predicts that

$$\|I + A + B\| = \lim_{n \rightarrow \infty} \lambda_{\max}(M_n) \approx 2.950.$$

However, we can in no way rule out that  $\|I + A + B\| = 3$ , i.e. that  $F$  is amenable. In the same way, we get in Case 2, ( $T = A + B + A^{-1} + B^{-1}$ ) the precise lower bound

$$\|A + A^{-1} + B + B^{-1}\| \geq \lambda_{\max}(M_{24}) = 3.60613$$

and that most likely we have

$$\|A + A^{-1} + B + B^{-1}\| \geq \alpha_{23} + \alpha_{24} = 3.68189.$$

Moreover, by making a least squares fitting of the 17 numbers

$$\lambda_{\max}(M_n), \quad 8 \leq n \leq 24$$

to a function of the form

$$f(n) = a - b(n - c)^{-d}, \quad n = 8, \dots, 24$$

we found the values of  $a, b, c, d$  to be

$$a = 3.870 \quad b = 1.612 \quad c = 0.573 \quad d = 0.480.$$

In particular, this extrapolation method predicts that

$$\|A + A^{-1} + B + B^{-1}\| = \lim_{n \rightarrow \infty} \lambda_{\max}(M_n) \approx 3.870$$

but again we cannot rule out that  $F$  is amenable.

In [7], Burillo, Cleary and Wiest used probabilistic methods to estimate the moments  $m_n$  in Case 2 ( $T = A + A^{-1} + B + B^{-1}$ ) for  $n = 10, 20, \dots, 160$ . In their notation,  $L = 2n$  and  $m_n = 4^L \hat{p}(L)$ . They found that (see Table 1 in [7])

$$m_{160}^{1/320} = 4\hat{p}(320)^{1/320} \approx 4 \cdot 0.9003 = 3.6012$$

and

$$\left(\frac{m_{160}}{m_{150}}\right)^{1/20} = 4 \left(\frac{\hat{p}(320)}{\hat{p}(300)}\right)^{1/20} \approx 4 \cdot 0.9161 = 3.6644.$$

Since  $m_n/m_{n-1}$  is increasing slowly for  $151 \leq n \leq 160$  we also have

$$\left(\frac{m_{160}}{m_{159}}\right)^{1/2} \approx \left(\frac{m_{160}}{m_{150}}\right)^{1/20} \approx 3.6644.$$

Their estimates are based on random samples of words in the generators, and are therefore not precise lower bound for  $\|A + A^{-1} + B + B^{-1}\|$ . In comparison, we found an exact lower bound 3.60613 of the norm based only on  $(m_n)_{1 \leq n \leq 24}$  and a very likely lower bound 3.68189 based on the same list of moments. In [13], Elder, Rechnitzer and Wong also worked on estimating the norm  $\|A + A^{-1} + B + B^{-1}\|$ . They found the lower bound

$$\|A + A^{-1} + B + B^{-1}\| \geq 3.55368$$

based on computing numerically the largest eigenvalue of the adjacency matrix for the Cayley graph of  $F$  restricted to balls in  $F$  of size  $N \leq 10^7$ , which corresponds to consider elements of  $F$  of distance up to 14 from the identity element. Moreover, they computed Cohen's cogrowth coefficients  $\gamma_{2n}$  for  $n = 1, 2, \dots, 11$  (cf. Table 3

in [13]). Note that  $\gamma_{2n}$  is equal to our “reduced” number  $\eta_n$  (cf. Section 3). In the notation of [13],  $\eta_n = p_{2n}$ , and  $m_n = r_{2n}$ , and they use a different method (based on power series) to pass from the  $(\eta_n)$ -series to the  $(m_n)$ -series.

### 5. ESTIMATING SPECTRAL DISTRIBUTION MEASURES

In this section, we will estimate the spectral distribution measures  $\mu_{\tilde{h}}$  for  $h = I + A + B$  (Case 1) and  $h = A + A^{-1} + B + B^{-1}$  (Case 2), based on the moment sequences listed in Section 3. This is done by computing the (possibly signed) measures  $\nu_N$  given by  $d\nu_N(t) = \rho_N(t)dt$  where  $\rho_N$  is the unique polynomial on  $\mathbb{R}$  of degree  $2N$  for which

$$(43) \quad \int_J t^n d\mu_{\tilde{h}}(t) = \int_J t^n d\nu_N(t), \quad 0 \leq n \leq 2N$$

where  $J = [-3, 3]$  in case 1, and  $J = [-4, 4]$  in case 2. We do not know, whether  $\mu_{\tilde{h}}$  has a density with respect to the Lebesgue measure, but if it has a density  $f$  and if  $f$  happens to be in  $L^2(J, dt)$ , then  $\rho_N$  is simply the orthogonal projection of  $f$  onto the subspace of  $L^2(J, dt)$  spanned by  $1, t, t^2, \dots, t^{2N}$ . Let  $(P_n(t))_{n=0}^\infty$  be the sequence of Legendre polynomials. Then it is well known that the sequence  $\left(\sqrt{n + \frac{1}{2}} P_n(t)\right)_{n=0}^\infty$  form an orthonormal basis for  $L^2([-1, 1], dt)$ . Hence

$$p_n(t) := \sqrt{\frac{n + \frac{1}{2}}{q + 1}} P_n\left(\frac{t}{q + 1}\right), \quad n \in \mathbb{N}_0$$

form an orthonormal basis for  $L^2(J, dt)$ , where  $q = 2$  in case 1 and  $q = 3$  in case 2. Hence in the case  $\mu_{\tilde{h}}$  has density  $f \in L^2(J, dt)$ ,  $\rho_N$  is the orthogonal projection of  $f$  onto  $\text{span}\{1, t, t^2, \dots, t^{2N}\}$ , i.e.

$$(44) \quad \rho_N(t) = \sum_{n=0}^{2N} \langle f, p_n \rangle p_n(t)$$

where

$$\langle f, p_n \rangle = \int_{-(q+1)}^{q+1} f(s) p_n(s) ds$$

Note that (44) can also be written as

$$(45) \quad \rho_N(t) = \sum_{n=0}^{2N} \left( \int_{-(q+1)}^{q+1} p_n(s) d\mu_{\tilde{h}}(s) \right) p_n(t).$$

The latter formula also makes sense if  $\mu_{\tilde{h}}$  does not have an  $L^2$ -density, and it is not hard to check that (45) provides the unique solution to (43) also when  $\mu_{\tilde{h}}$  does not have an  $L^2$ -density. Based on our moment calculations we can compute the polynomials  $\rho_N(t)$  for  $1 \leq N \leq 37$  in case 1 and for  $1 \leq N \leq 24$  in case 2. Since  $\mu_{\tilde{h}}$  is a symmetric measure, all the odd terms in (45) are zero. Hence  $\rho_N$  is an even polynomial of degree at most  $2N$ .

Case 1: In Fig. 4 we have for  $h = I + A + B$  plotted  $\rho_{37}$  for  $0 \leq t \leq 3$  together with the corresponding “free” density (cf. Section 1) and in Figs. 5-6 we have plotted first  $\rho_{36}$  and  $\rho_{37}$  in the tail interval  $[2\sqrt{2}, 3]$  and next  $\frac{1}{2}(\rho_{36} + \rho_{37})$  in the same interval. The reason is that the plot in Fig. 5 shows that for  $t$  close to 3,  $\rho_{36}$  and  $\rho_{37}$  oscillates around zero with opposite signs, and therefore we expect that  $\frac{1}{2}(\rho_{36}(t) + \rho_{37}(t))dt$  gives a better approximation to the measure  $\mu_{\tilde{h}}$ . Figure 6 indicates, that  $\mu_{\tilde{h}}$  has

very little mass on the interval  $[2.9, 3.0]$ , and hence  $\|h\| = \max(\sup(\mu_{\tilde{h}}))$  could be any number in the interval  $[2.9, 3.0]$ , including the number 2.95 found in Section 4 by an extrapolation argument. Recall from the end of Section 2 that  $\mu_{h^*h}$  is the image measure of  $\mu_{\tilde{h}}$  by the map  $t \mapsto t^2$ . Hence Figs. 4-6 can also be used to compute an approximation to the spectral density of  $(I + A + B)^*(I + A + B)$ .

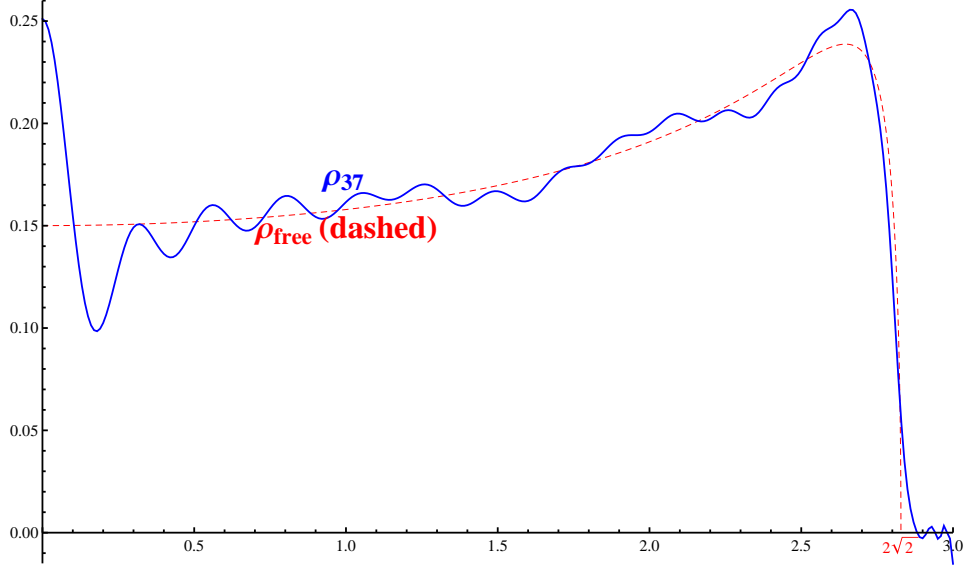


FIGURE 4. Estimating the Lebesgue density for  $\mu_{\tilde{h}}$ , where  $h = I + A + B$

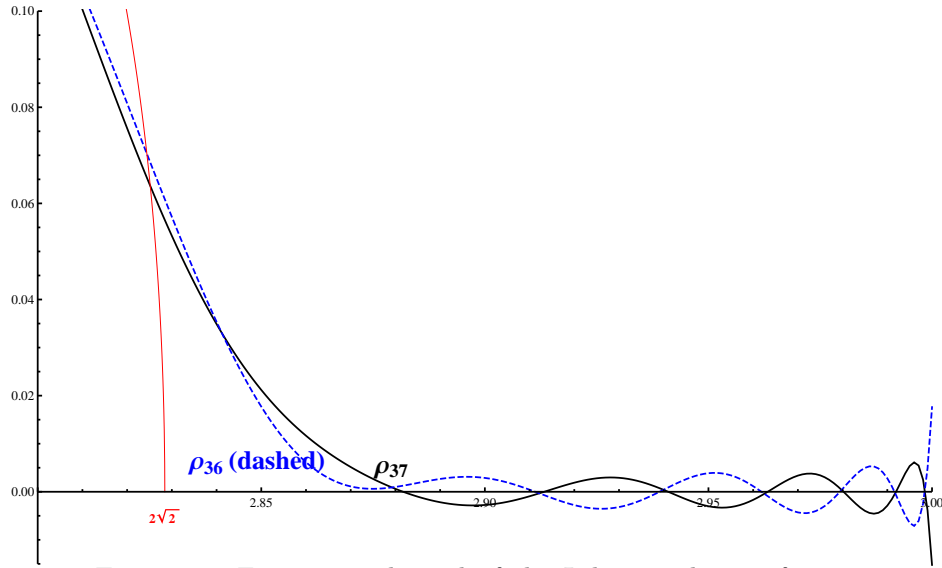


FIGURE 5. Estimating the tail of the Lebesgue density for  $\mu_{\tilde{h}}$ , where  $h = I + A + B$

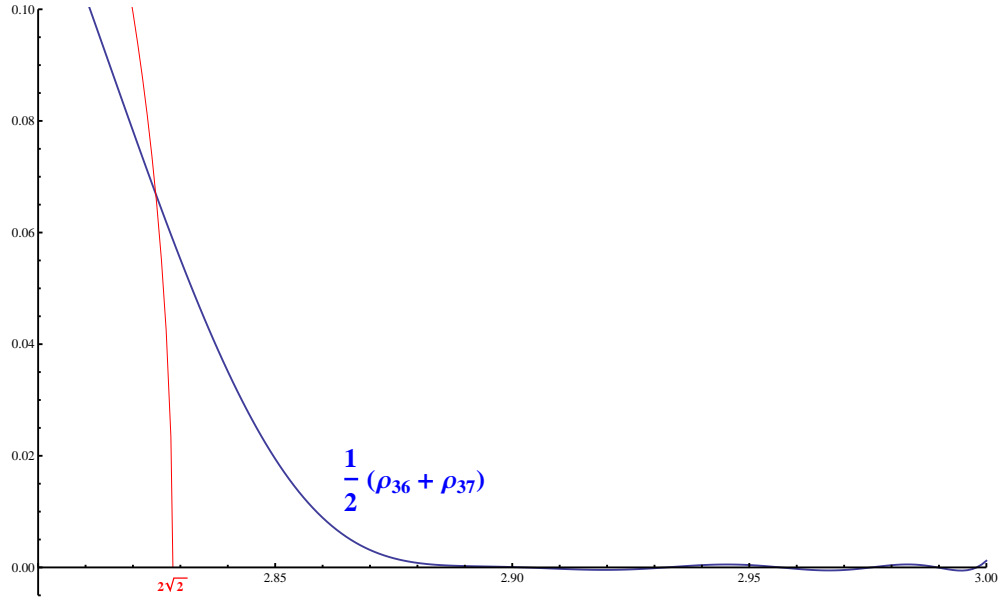


FIGURE 6. A better estimate of the Lebesgue density for  $\mu_{\tilde{h}}$ , where  $h = I + A + B$

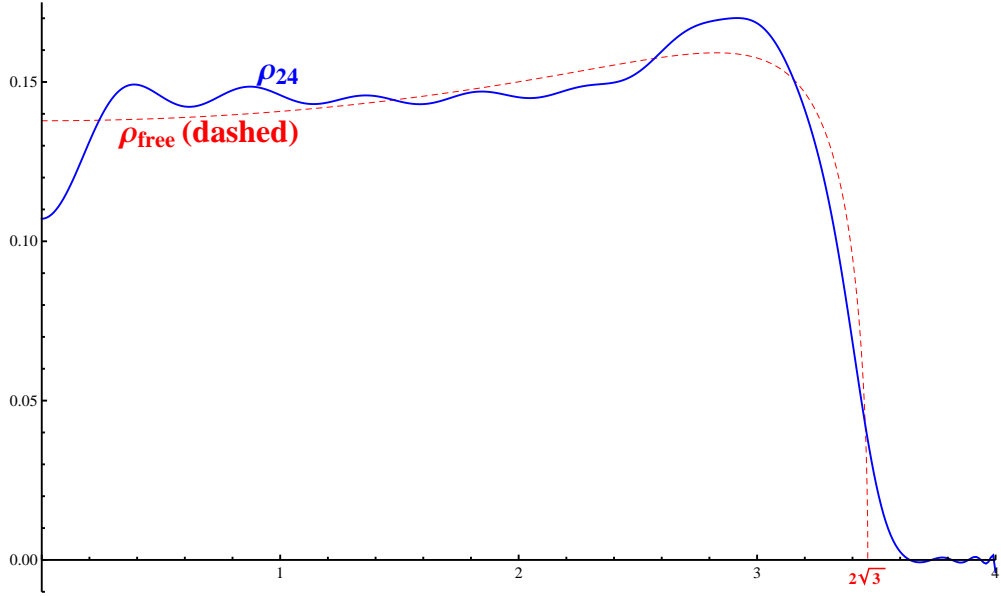


FIGURE 7. Estimating the Lebesgue density for  $\mu_{\tilde{h}} = \mu_h$ , where  $h = A + B + A^{-1} + B^{-1}$

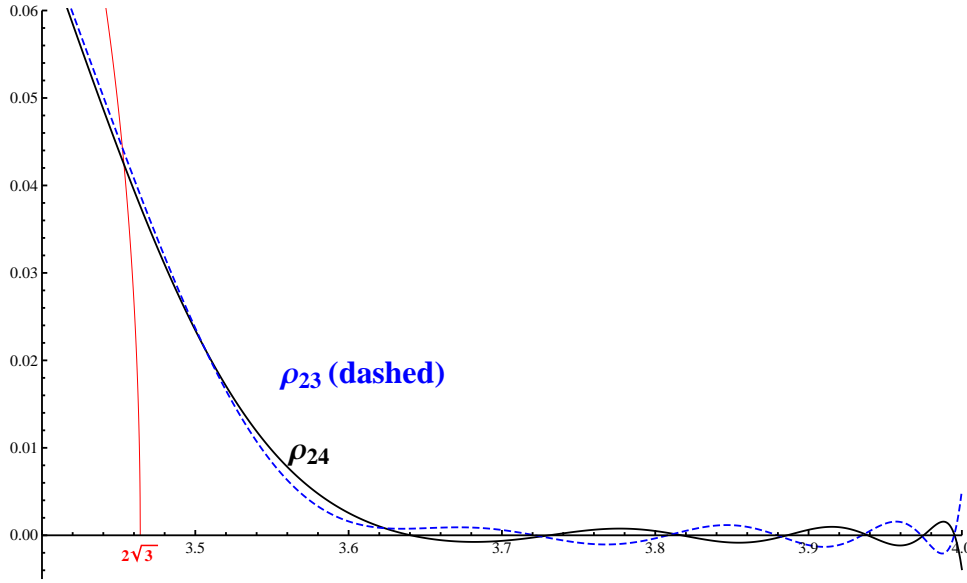


FIGURE 8. Estimating the tail of the Lebesgue density for  $\mu_{\tilde{h}} = \mu_h$ , where  $h = A + B + A^{-1} + B^{-1}$

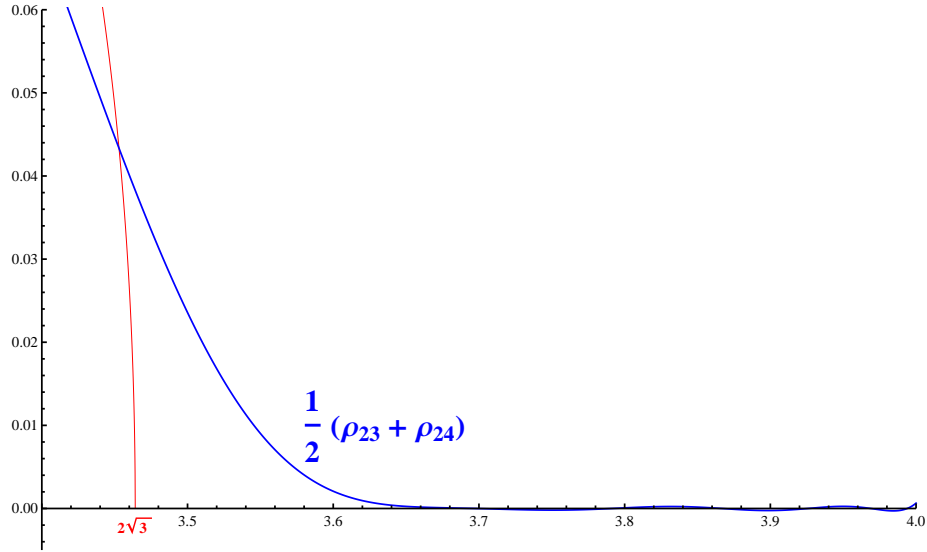


FIGURE 9. A better estimate of the Lebesgue density for  $\mu_{\tilde{h}} = \mu_h$ , where  $h = A + B + A^{-1} + B^{-1}$

Case 2: In Fig. 7 we have for  $h = A + A^{-1} + B + B^{-1}$  plotted  $\rho_{24}$  for  $0 \leq t \leq 4$  together with the corresponding “free” density, and in Figs. 8-9 we have plotted  $\rho_{23}$ ,  $\rho_{24}$  and  $\frac{1}{2}(\rho_{23} + \rho_{24})$  in the tail interval  $[2\sqrt{3}, 4]$ . Again  $\frac{1}{2}(\rho_{23}(t) + \rho_{24}(t))dt$  appear to give the best approximation to the measure  $\mu_{\tilde{h}}$  in the tail region, and it shows that  $\mu_{\tilde{h}}$  must have very little mass in the interval  $[3.7, 4.0]$ . Hence  $\|h\| =$



$\max(\sup(\mu_{\tilde{h}}))$  could be any number in the interval  $[3.7, 4.0]$  including the number 3.87 found in Section 4 by an extrapolation argument. Note finally, that in the case of  $h = A + A^{-1} + B + B^{-1}$ ,  $\mu_h = \mu_{\tilde{h}}$ . To see this, we just have to check that  $\mu_h$  and  $\mu_{\tilde{h}}$  have the same moments. The even moments coincide because  $h = h^*$  and therefore

$$m_{2n}(\tilde{h}) = \tau((h^*h)^n) = \tau(h^{2n}) = m_{2n}(h), \quad n \in \mathbb{N}_0.$$

Moreover,

$$m_{2n+1}(\tilde{h}) = 0 = m_{2n+1}(h), \quad n \in \mathbb{N}_0,$$

where the first equality follows because  $\mu_{\tilde{h}}$  is symmetric and the second equality follows because no word of odd length in  $A, A^{-1}, B, B^{-1}$  can represent the identity in  $F$ .

## 6. RELATIONS BETWEEN $\|h_n\|_2^2$ , $\xi_n$ , $\eta_n$ , $\zeta_n$ AND $m_n$

**6.1. The polynomials  $Q_n$  with constant  $q \in \mathbb{N}$ .** Let  $q \in \mathbb{N}$  be a fixed natural number. Define the polynomials  $(Q_n)_{n \in \mathbb{N}}$  in  $\mathbb{C}[t]$  recursively by

$$(46) \quad \begin{aligned} Q_1(t) &= t \\ Q_2(t) &= t^2 - (q+1) \\ Q_{n+1}(t) &= tQ_n(t) - qQ_{n-1}(t), \quad n \geq 2. \end{aligned}$$

Since  $Q_n$  is an even polynomial for  $n$  even and an odd polynomial for  $n$  odd, there exists polynomials  $(Q_m^{(1)})_{m \geq 1}$  and  $(Q_m^{(2)})_{m \geq 0}$  in  $\mathbb{C}[t]$  such that

$$(47) \quad \begin{aligned} Q_{2m}(t) &= Q_m^{(1)}(t^2) \quad m \geq 1 \\ Q_{2m+1}(t) &= tQ_m^{(2)}(t^2) \quad m \geq 0. \end{aligned}$$

Next we write the  $Q_n$  polynomials in terms of Chebyshev polynomials.

**Proposition 6.1.** *Let  $(T_n)_{n \in \mathbb{N}}$  and  $(U_n)_{n \in \mathbb{N}}$  be the Chebyshev polynomials of first and second kind. Then*

$$(i) \quad Q_n(t) = \left( \frac{2}{q} T_n \left( \frac{t}{2\sqrt{q}} \right) + \frac{q-1}{q} U_n \left( \frac{t}{2\sqrt{q}} \right) \right) q^{n/2}, \quad n \in \mathbb{N}, \quad t \in \mathbb{R}.$$

$$(ii) \quad Q_{2n}(t) - (q-1) \sum_{k=1}^{n-1} Q_{2k}(t) = (q-1) + 2T_{2n} \left( \frac{t}{2\sqrt{q}} \right) q^n, \quad n \in \mathbb{N}, \quad t \in \mathbb{R}.$$

*Proof.* Let  $Q_0(t) := \frac{q+1}{q}$ . Then the recursion formula

$$Q_{n+1}(t) = tQ_n(t) - qQ_{n-1}(t)$$

holds for all  $n \geq 1$ . Letting

$$q_n(t) := Q_n(2\sqrt{q}t)q^{-n/2}, \quad n \geq 0, \quad t \in \mathbb{R},$$

we get

$$\begin{aligned} q_0(t) &= \frac{q+1}{q} \\ q_1(t) &= 2t \\ q_{n+1}(t) &= 2tq_n(t) - q_{n-1}(t), \quad n \geq 1. \end{aligned}$$

The Chebyshev polynomials also satisfy an identical recursion formula:

$$\begin{aligned} T_{n+1}(t) &= 2tT_n(t) - T_{n-1}(t), \quad n \geq 1, \\ U_{n+1}(t) &= 2tU_n(t) - U_{n-1}(t), \quad n \geq 1. \end{aligned}$$

Hence, if we can choose constants  $\alpha, \beta \in \mathbb{R}$  such that

$$q_0(t) = \alpha T_0(t) + \beta U_0(t) \quad q_1(t) = \alpha T_1(t) + \beta U_1(t),$$

then  $q_n(t) = \alpha T_n(t) + \beta U_n(t)$  would hold for all  $n \geq 0$ . Indeed,  $\alpha = 2/q$  and  $\beta = (q-1)/q$  is the only solution since  $T_0(t) = 1$ ,  $T_1(t) = t$ ,  $U_0(t) = 1$ ,  $U_1(t) = 2t$ . This proves (i) because

$$(48) \quad Q_n(t) = q_n\left(\frac{t}{2\sqrt{q}}\right)q^{n/2}, \quad n \geq 0, t \in \mathbb{R}.$$

The proof of (ii) is an adaptation from the proof of formula (2.6) in [18]. From Eq. (2) p. 184 in [20], the Chebyshev polynomials satisfy

$$T_n(\cos \theta) = \cos n\theta, \quad U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}.$$

It follows that

$$U_n(t) - U_{n-2}(t) = 2T_n(t), \quad n \geq 2,$$

for  $-1 < t < 1$  and hence for all  $t \in \mathbb{R}$  because we are dealing with polynomials.

Since  $q_n(t) = \frac{2}{q}T_n(t) + \frac{q-1}{q}U_n(t)$ ,  $t \in \mathbb{R}$ ,  $n \geq 0$ , we get

$$q_n(t) - q_{n-2}(t) = 2T_n(t) - \frac{2}{q}T_{n-2}(t), \quad t \in \mathbb{R}, n \geq 2.$$

From this we get

$$\sum_{k=1}^n (q_{2k}(t) - q_{2k-2}(t))q^{k-n} = 2T_{2n}(t) - \frac{2}{q^n}T_0(t) = 2T_{2n}(t) - \frac{2}{q^n}.$$

Simplifying,

$$q_{2n}(t) - (q-1) \sum_{k=1}^{n-1} \frac{q_{2k}(t)}{q^{n-k}} - \frac{q_0(t)}{q^{n-1}} = 2T_{2n}(t) - \frac{2}{q^n}.$$

Therefore

$$q_{2n}(t) - (q-1) \sum_{k=1}^{n-1} \frac{q_{2k}(t)}{q^{n-k}} = 2T_{2n}(t) + \frac{q-1}{q^n}.$$

The last equation together with Eq. (48) yields (ii).  $\square$

**6.2. The group ring sequences  $(h_n)$ ,  $(k_n)$ .** Let  $\Gamma$  be a discrete group, let  $Y \subset \Gamma$  be a finite set with  $|Y| = q+1$  elements ( $q \in \mathbb{N}$ ), and let  $h = \sum_{s \in Y} s$  as in Section 3. Define  $Q_n$ ,  $Q_n^{(1)}$ ,  $Q_n^{(2)}$  as in Eqs. (46) and (47) for this value of  $q$ . Define the sequences  $(h_n)_{n \in \mathbb{N}}$ ,  $(k_n)_{n \in \mathbb{N}}$  in the group ring  $\mathbb{C}\Gamma$  of  $\Gamma$  by

$$(49) \quad \begin{aligned} E_n &:= \{(s_1, \dots, s_n) \in Y^n \mid s_1 \neq s_2 \neq \dots \neq s_n\}. \\ h_n &:= \sum_{(s_1, \dots, s_n) \in E_n} s_1(s_2^{-1}s_3 \cdots s_{n-1}^{-1}s_n) \quad (n \text{ odd}) \\ h_n &:= \sum_{(s_1, \dots, s_n) \in E_n} s_1^{-1}s_2 \cdots s_{n-1}^{-1}s_n \quad (n \text{ even}). \end{aligned}$$

$$(50) \quad \begin{aligned} k_n &:= \sum_{(s_1, \dots, s_n) \in E_n} s_1^{-1}(s_2s_3^{-1} \cdots s_{n-1}s_n^{-1}) \quad (n \text{ odd}) \\ k_n &:= \sum_{(s_1, \dots, s_n) \in E_n} s_1s_2^{-1} \cdots s_{n-1}s_n^{-1} \quad (n \text{ even}). \end{aligned}$$

In all cases,  $s_1 \neq s_2 \neq \dots \neq s_n$  means  $s_i \neq s_{i+1}$  ( $1 \leq i \leq n-1$ ), and  $Y^{-1} := \{x^{-1} : x \in Y\}$ . Notice that  $h_1 = h$  and  $k_1 = h^*$ . In fact  $h_n^* = k_n$  for  $n$  odd, and for  $n$  even  $h_n^* = h_n$  and  $k_n^* = k_n$ .

**Theorem 6.2.** *Let  $(h_n)_{n \in \mathbb{N}}$ ,  $(k_n)_{n \in \mathbb{N}}$  be the sequences defined in Eqs. (49) and (50). Then in the algebra  $M_2(\mathbb{C}\Gamma)$  of  $2 \times 2$  matrices over  $\mathbb{C}\Gamma$  we have*

$$(i) \begin{pmatrix} 0 & k_n \\ h_n & 0 \end{pmatrix} = Q_n \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix} \text{ for } n \text{ odd.}$$

$$(ii) \begin{pmatrix} h_n & 0 \\ 0 & k_n \end{pmatrix} = Q_n \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix} \text{ for } n \text{ even.}$$

These formulas can also be expressed in the group ring  $\mathbb{C}(\Gamma)$  as

$$(iii) \begin{aligned} h_{2m+1} &= hQ_m^{(2)}(h^*h), \\ k_{2m+1} &= h^*Q_m^{(2)}(hh^*) \quad m \geq 0. \end{aligned}$$

$$(iv) \begin{aligned} h_{2m} &= Q_m^{(1)}(h^*h), \\ k_{2m} &= Q_m^{(1)}(hh^*) \quad m \geq 1. \end{aligned}$$

*Proof.* We start by proving (i) and (ii) by induction in  $n \in \mathbb{N}$ . Note first that (i) holds for  $n = 1$  and that (ii) holds for  $n = 2$  because

$$h_2 = \left( \sum_{s_1 \in Y} s_1^{-1} \right) \left( \sum_{s_2 \in Y} s_2 \right) - |Y|e = h^*h - (q+1)e,$$

and similarly  $k_2 = hh^* - (q+1)e$ . We next prove the following 4 recursion formulas.

$$(51) \quad \begin{aligned} h_{n+1} &= hh_n - qh_{n-1}, & n \geq 2, n \text{ even} \\ k_{n+1} &= h^*k_n - qk_{n-1}, & n \geq 2, n \text{ even.} \end{aligned}$$

$$(52) \quad \begin{aligned} h_{n+1} &= h^*h_n - qh_{n-1}, & n \geq 3, n \text{ odd} \\ k_{n+1} &= hk_n - qk_{n-1}, & n \geq 3, n \text{ odd.} \end{aligned}$$

To prove the first formula in Eq. (51) note that  $n+1$  is odd. Hence

$$\begin{aligned} h_{n+1} &= \sum_{(s_0, \dots, s_n) \in E_{n+1}} s_0(s_1^{-1}s_2 \dots s_{n-1}^{-1}s_n) \\ &= \sum_{(s_1, \dots, s_n) \in E_n} \sum_{s_0 \in Y \setminus \{s_1\}} s_0(s_1^{-1}s_2 \dots s_{n-1}^{-1}s_n) \\ &= \sum_{(s_1, \dots, s_n) \in E_n} \left( \sum_{s_0 \in Y} s_0(s_1^{-1}s_2 \dots s_{n-1}^{-1}s_n) - s_1(s_1^{-1}s_2 \dots s_{n-1}^{-1}s_n) \right) \\ &= \left( \sum_{s_0 \in Y} s_0 \right) \left( \sum_{(s_1, \dots, s_n) \in E_n} s_1^{-1}s_2 \dots s_{n-1}^{-1}s_n \right) - \\ &\quad \sum_{(s_2, \dots, s_n) \in E_{n-1}} \left( \sum_{s_1 \in Y \setminus \{s_2\}} s_1 s_1^{-1} \right) (s_2 \dots s_{n-1}^{-1}s_n) \\ &= hh_n - qh_{n-1}. \end{aligned}$$

The second formula in Eq. (51) follows from this by replacing  $Y$  with  $Y^{-1}$ . The two formulas in Eq. (52) can be proven in exactly the same way noticing that in this case  $n+1$  is even. The formulas in Eqs. (51) and (52) can be rewritten as

$$\begin{pmatrix} 0 & k_{n+1} \\ h_{n+1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix} \begin{pmatrix} h_n & 0 \\ 0 & k_n \end{pmatrix} - q \begin{pmatrix} 0 & k_{n-1} \\ h_{n-1} & 0 \end{pmatrix}$$

for  $n \geq 2$ ,  $n$  even; and

$$\begin{pmatrix} h_{n+1} & 0 \\ 0 & k_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix} \begin{pmatrix} 0 & k_n \\ h_n & 0 \end{pmatrix} - q \begin{pmatrix} h_{n-1} & 0 \\ 0 & k_{n-1} \end{pmatrix}$$

for  $n \geq 3$ ,  $n$  odd. By induction in  $n$  we get

$$Q_{n+1}(t) = tQ_n(t) - qQ_{n-1}(t), \quad n \geq 2,$$

once we rewrite the above two matrix equations in terms of (i) and (ii), which is precisely the definition of  $Q_n$  given in Formula (46). Hence (i) and (ii) hold. Since

$$\begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix}^2 = \begin{pmatrix} h^*h & 0 \\ 0 & hh^* \end{pmatrix},$$

we have by Eq. (47)

$$\begin{aligned} \begin{pmatrix} 0 & k_{2m+1} \\ h_{2m+1} & 0 \end{pmatrix} &= Q_{2m+1} \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix} Q_m^{(2)} \begin{pmatrix} h^*h & 0 \\ 0 & hh^* \end{pmatrix} \\ &= \begin{pmatrix} 0 & h^*Q_m^{(2)}(hh^*) \\ hQ_m^{(2)}(h^*h) & 0 \end{pmatrix} \end{aligned}$$

for  $m \geq 0$ , proving (iii). Similarly for  $m \geq 1$ , we have

$$\begin{pmatrix} h_{2m} & 0 \\ 0 & k_{2m} \end{pmatrix} = Q_{2m} \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix} = \begin{pmatrix} Q_m^{(1)}(h^*h) & 0 \\ 0 & Q_m^{(1)}(hh^*) \end{pmatrix}$$

proving (iv).  $\square$

**Remark 6.3.** If  $Y = Y^{-1}$  then  $h = h^*$  and  $h_n = k_n = Q_n(h)$  for all  $n \in \mathbb{N}$ .

**Proposition 6.4.** For all  $n \in \mathbb{N}$  we have

$$(53) \quad h_n^* h_n = h_{2n} + (q+1)q^{n-1}e + (q-1) \sum_{i=1}^{n-1} q^{i-1} h_{2n-2i}.$$

*Proof.* For  $n = 1$  we have from the proof of Theorem 6.2 that  $h^*h = h_2 + (q+1)e$ . Consider now  $n \geq 2$ . If  $n$  is odd, then

$$h_n^* h_n = \sum_{\substack{(t_1, \dots, t_n) \in E_n \\ (s_1, \dots, s_n) \in E_n}} (t_n^{-1} t_{n-1} \cdots t_3^{-1} t_2) t_1^{-1} s_1 (s_2^{-1} s_3 \cdots s_{n-1}^{-1} s_n) = \sum_{k=0}^n a_k,$$

where  $a_k$  is obtained by summing over only those  $(s_1, \dots, s_n), (t_1, \dots, t_n) \in E_n$  for which  $(s_1, \dots, s_k) = (t_1, \dots, t_k)$  and  $s_{k+1} \neq t_{k+1}$  when  $0 \leq k \leq n-1$ , and  $(s_1, \dots, s_n) = (t_1, \dots, t_n)$  when  $k = n$ . If  $k = 0$  then  $s_1 \neq t_1$  and thus  $a_0 = h_{2n}$ . If  $k = n$  then  $(t_1, \dots, t_n) = (s_1, \dots, s_n)$  and thus  $a_n = |E_n|e = (q+1)q^{n-1}e$ . For  $1 \leq k \leq n-1$ ,  $k$  odd, we have

$$\begin{aligned} a_k &= \sum_{\substack{(t_{k+1}, \dots, t_n) \in E_{n-k} \\ (s_{k+1}, \dots, s_n) \in E_{n-k} \\ t_{k+1} \neq s_{k+1}}} \sum_{\substack{(t_1, \dots, t_k) \in E_k \\ (s_1, \dots, s_k) \in E_k \\ (t_1, \dots, t_k) = (s_1, \dots, s_k) \\ t_k \neq t_{k+1} \\ s_k \neq s_{k+1}}} (t_n^{-1} t_{n-1} \cdots t_{k+2}^{-1} t_{k+1}) (s_{k+1}^{-1} s_{k+2} \cdots s_{n-1}^{-1} s_n) \\ &= (q-1)q^{k-1} h_{2n-2k} \end{aligned}$$

because for fixed  $(t_{k+1}, \dots, t_n), (s_{k+1}, \dots, s_n) \in E_{n-k}$  with  $t_{k+1} \neq s_{k+1}$ ,  $(s_1, \dots, s_k) = (t_1, \dots, t_k)$  can be chosen in  $(q-1)q^{k-1}$  ways, namely  $s_k = t_k \in Y \setminus \{s_{k+1}, t_{k+1}\}$  can first be chosen in  $|Y| - 2 = q - 1$  ways, and next  $s_{k-1} = t_{k-1}$ ,  $s_{k-2} = t_{k-2}$  etc. can each be chosen in  $q = |Y| - 1$  ways. The same holds for  $k$  even and/or  $n$  even by obvious modifications of the above proof.  $\square$

**Proposition 6.5.** *We have*

$$(54) \quad \|h_n\|_2 = \|k_n\|_2, \quad n \in \mathbb{N}$$

*Proof.* For  $n$  odd,  $h_n^* = k_n$  by Theorem 6.2. Thus  $\|h_n\|_2 = \tau(h_n^* h_n)^{1/2} = \tau(h_n h_n^*)^{1/2} = \|k_n\|_2$ . For  $n$  even,  $n = 2m$ , Theorem 6.2 yields  $\|h_n\|_2^2 = \tau((Q_m^{(1)}(h^* h))^2) = \tau((Q_m^{(1)}(h h^*))^2) = \|k_n\|_2^2$  as  $\tau((h^* h)^j) = \tau((h^* h)^{j-1} (h^* h)) = \tau(h (h^* h)^{j-1} h^*) = \tau((h h^*)^j)$  for any  $j \in \mathbb{N}$ .  $\square$

**6.3. The integer sequences  $\xi_n, \eta_n, \zeta_n$ .** Define the group ring sequence  $(z_n)_{n \in \mathbb{N}}$  by

$$(55) \quad z_n := \sum_{(s_1, \dots, s_{2n}) \in \tilde{E}_{2n}} s_1^{-1} s_2 \cdots s_{2n-1}^{-1} s_{2n},$$

where  $\tilde{E}_k := \{(s_1, \dots, s_k) \in E_k \mid s_1 \neq s_k\}$ . We could say that  $z_n$  is the cyclic version of  $h_{2n}$  defined in Eq. (49). Recall that  $E_n \subset Y^n$ , where  $Y \subset \Gamma$  is a finite set with  $|Y| = q + 1$  elements. We assume  $q \geq 1$ . Define the number sequences  $(\xi_n)_{n \in \mathbb{N}}, (\eta_n)_{n \in \mathbb{N}}, (\zeta_n)_{n \in \mathbb{N}}$  by

$$(56) \quad \begin{aligned} \xi_n &:= \|h_n\|_2^2 - (q+1)q^{n-1} \\ \eta_n &:= \tau(h_{2n}) \\ \zeta_n &:= \tau(z_n). \end{aligned}$$

**Proposition 6.6.** *For  $n \in \mathbb{N}$ ,  $\xi_n, \eta_n, \zeta_n$  are integers and*

$$(57) \quad 0 \leq \zeta_n \leq \eta_n \leq \xi_n \leq 4q^{2n}.$$

*Proof.* The numbers  $\xi_n, \eta_n, \zeta_n$  are integers because  $h_n \in \mathbb{Z}\Gamma$  and  $\tau(x) \in \{0, 1\}$  for all  $x \in \Gamma$ . Since  $\tau(x) \geq 0$  for all  $x \in \Gamma$ , and  $h_n$  is a sum on  $E_{2n}$  while  $z_n$  is the same sum but on a subset of  $E_{2n}$ , we have  $\eta_n \geq \zeta_n \geq 0$ . By Proposition 6.4

$$\xi_n = \tau(h_n^* h_n) - (q+1)q^{n-1} = \eta_n + (q-1) \sum_{i=1}^{n-1} q^{i-1} \eta_{n-i} \geq \eta_n.$$

Finally,

$$\xi_n \leq \|h_n\|^2 = \tau(h_n^* h_n) \leq \sum_{s, t \in E_n} 1 = |E_n|^2 = ((q+1)q^{n-1})^2 = \left(\frac{q+1}{q}\right)^2 q^{2n} \leq 4q^{2n}.$$

$\square$

**Proposition 6.7.**

$$\tau(h_{2n}) = \tau(z_n) + (q-1) \sum_{k=1}^{n-1} q^{k-1} \tau(z_{n-k})$$

*Proof.* We have

$$h_{2n} = \sum_{(s_1, \dots, s_{2n}) \in E_{2n}} s_{2n}^{-1} s_{2n-1} \cdots s_2^{-1} s_1 = \sum_{k=0}^n b_k$$

where

$$b_k = \sum_{(s_1, \dots, s_{2n}) \in E_{2n}^{(k)}} s_{2n}^{-1} s_{2n-1} \cdots s_2^{-1} s_1$$

and where for  $1 \leq k \leq n-1$ ,  $E_{2n}^{(k)}$  denotes the subset of  $(s_1, \dots, s_{2n}) \in E_{2n}$  for which

$$s_1 = s_{2n}, s_2 = s_{2n-1}, \dots, s_k = s_{2n-k+1}, s_{k+1} \neq s_{2n-k}$$

and for  $k \in \{0, n\}$ :

$$E_{2n}^{(0)} = \{(s_1, \dots, s_{2n}) \in E_{2n} \mid s_1 \neq s_{2n}\} = \tilde{E}_{2n},$$

$$E_{2n}^{(n)} = \{(s_1, \dots, s_{2n}) \in E_{2n} \mid s_i = s_{2n+1-i}, i = 1, \dots, n\}.$$

Clearly  $b_0 = z_n$ . Moreover  $b_n = 0$  because  $E_{2n}^{(n)} = \emptyset$  as  $s_n \neq s_{n+1}$ . For  $1 \leq k \leq n-1$  ( $k$  odd) we can write

$$b_k = \sum u_1^{-1} u_2 \cdots u_k^{-1} (s_{k+1} s_{k+2}^{-1} \cdots s_{2n-k}) u_k \cdots u_2^{-1} u_1$$

where the summation is over all  $(s_{k+1}, s_{k+2}, \dots, s_{2n-k}) \in \tilde{E}_{2n-2k}$  and  $(u_1, u_2, \dots, u_k) \in E_k$  for which  $u_k \notin \{s_{k+1}, s_{2n-k}\}$ .

For fixed  $(s_{k+1}, s_{k+2}, \dots, s_{2n-k})$  there are exactly  $(q-1)q^{k-1}$  choices of  $(u_1, \dots, u_k)$ , namely first  $u_k$  can be chosen in  $|Y| - 2 = q - 1$  ways because  $s_{k+1} \neq s_{2n-k}$  and next each of  $u_{k-1}, u_{k-2}, \dots, u_1$  can be chosen in  $|Y| - 1 = q$  ways. Since

$$\tau(u_1^{-1} u_2 \cdots u_k^{-1} (s_{k+1} s_{k+2}^{-1} \cdots s_{2n-k}) u_k \cdots u_2^{-1} u_1) = \tau(s_{k+1} s_{k+2}^{-1} \cdots s_{2n-k})$$

it follows that

$$\begin{aligned} \tau(b_k) &= (q-1)q^{k-1} \sum_{(s_{k+1}, \dots, s_{2n-k}) \in \tilde{E}_{2n-2k}} \tau(s_{k+1} s_{k+2}^{-1} \cdots s_{2n-k}) \\ &= (q-1)q^{k-1} \tau(z_{n-k}). \end{aligned}$$

The same formula holds for  $k$  even ( $1 \leq k \leq n-1$ ) by an obvious modification of the proof. This proves the proposition.  $\square$

**Proposition 6.8.** *For  $n \geq 1$  we have*

$$\begin{aligned} (i) \quad \xi_n &= \eta_n + (q-1) \sum_{i=1}^{n-1} q^{i-1} \eta_{n-i} \\ (ii) \quad \eta_n &= \xi_n - (q-1) \sum_{i=1}^{n-1} \xi_i. \end{aligned}$$

Similarly,

$$\begin{aligned} (iii) \quad \eta_n &= \zeta_n + (q-1) \sum_{i=1}^{n-1} q^{i-1} \zeta_{n-i} \\ (iv) \quad \zeta_n &= \eta_n - (q-1) \sum_{i=1}^{n-1} \eta_i. \end{aligned}$$

*Proof.* Consider the power series,

$$A(t) := \sum_{n=1}^{\infty} \xi_n t^n, \quad B(t) := \sum_{n=1}^{\infty} \eta_n t^n, \quad C(t) := \sum_{n=1}^{\infty} \zeta_n t^n.$$

By Proposition 6.6, they are all convergent for all  $t \in \mathbb{C}$  with  $|t| < \frac{1}{q^2}$ . We have already seen that (i) follows from Proposition 6.4. By (i) we have for  $|t| < \frac{1}{q^2}$ :

$$A(t) = B(t) \left( 1 + (q-1) \sum_{k=0}^{\infty} q^k t^{k+1} \right) = B(t) \frac{1-t}{1-qt}.$$

Hence for  $|s| < \frac{1}{q^2}$ , we have

$$B(t) = A(t) \frac{1-qt}{1-t} = A(t) \left( 1 - (q-1) \sum_{k=1}^n t^k \right)$$

By comparing the coefficients of  $t^n$  in the power series expansion of  $B(t)$  and of  $A(t) \frac{1-qt}{1-t}$  we get (ii). Note next that (iii) follows from Proposition 6.7. Hence, as in the proof of (i)  $\Rightarrow$  (ii) we get

$$B(t) = C(t) \frac{1-t}{1-qt} \quad \text{and} \quad C(t) = \frac{1-qt}{1-t} B(t)$$

which implies (iv).  $\square$

**6.4. The symmetric measure  $\mu_{\tilde{h}}$ .** Let  $Y \subset \Gamma$  be a finite subset with  $|Y| = q+1$  elements and let as before

$$h := \sum_{s \in Y} s \in \mathbb{C}\Gamma$$

and

$$(58) \quad \tilde{h} := \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sum_{s \in S} s^{-1} \\ \sum_{s \in S} s & 0 \end{pmatrix} \in M_2(\mathbb{C}\Gamma).$$

Recall that the trace  $\tilde{\tau} = \tau \otimes \tau_2$  on  $M_2(\mathbb{C}\Gamma)$  was defined in Eq. (10). Let  $\mu_{\tilde{h}}$  be the spectral measure of  $\tilde{h}$  on the interval  $[-(q+1), q+1]$  with respect to the trace  $\tilde{\tau}$ , i.e. the unique probability measure on  $[-(q+1), q+1]$  satisfying

$$\tilde{\tau}(\tilde{h}^n) = \int_{-(q+1)}^{q+1} t^n d\mu_{\tilde{h}}, \quad n \in \mathbb{N}_0.$$

(See Section 2). Since

$$\tilde{h}^{2m} = \begin{pmatrix} (h^*h)^m & 0 \\ 0 & (hh^*)^m \end{pmatrix} \quad m \geq 0$$

and

$$\tilde{h}^{2m+1} = \begin{pmatrix} 0 & h^*(hh^*)^m \\ h(h^*h)^m & 0 \end{pmatrix} \quad m \geq 0$$

we have

$$\int_{-(q+1)}^{q+1} t^{2m} d\mu_{\tilde{h}}(t) = \frac{\tau((h^*h)^m) + \tau((hh^*)^m)}{2} = \tau((h^*h)^m), \quad m \geq 0,$$

and

$$\int_{-(q+1)}^{q+1} t^{2m+1} d\mu_{\tilde{h}}(t) = 0, \quad m \geq 0.$$

The latter condition implies via Riesz representation theorem that  $\mu_{\tilde{h}}$  is symmetric. i.e.  $\mu_{\tilde{h}} = \check{\mu}_{\tilde{h}}$ , where  $\check{\mu}_{\tilde{h}}$  is the image measure of  $\mu_{\tilde{h}}$  with respect to the map  $t \mapsto -t$ . (This could also be shown algebraically). Let

$$\begin{aligned} (59) \quad m_n &:= \tau((h^*h)^n) \\ &= \int_{-(q+1)}^{q+1} t^{2n} d\mu_{\tilde{h}}(t). \end{aligned}$$

Notice that  $m_0 = 1$  and  $m_1 = q + 1$ .

**Proposition 6.9.** *For  $n \in \mathbb{N}$ , let  $\eta_n, \zeta_n$  be as in Eq. (56), and  $T_n, U_n$  be the Chebyshev polynomials of first and second kind. Then*

$$\begin{aligned} (i) \quad \eta_n &= q^n \int_{-(q+1)}^{q+1} \frac{2}{q} T_{2n} \left( \frac{t}{2\sqrt{q}} \right) + \frac{q-1}{q} U_{2n} \left( \frac{t}{2\sqrt{q}} \right) d\mu_{\tilde{h}}(t) \\ (ii) \quad \zeta_n &= (q-1) + 2q^n \int_{-(q+1)}^{q+1} T_{2n} \left( \frac{t}{2\sqrt{q}} \right) d\mu_{\tilde{h}}(t). \end{aligned}$$

*Proof.* Since  $\tau((h^*h)^m) = \tilde{\tau}(\tilde{h}^{2m}) = \int_{-(q+1)}^{q+1} t^{2m} d\mu_{\tilde{h}}(t)$ ,  $m \geq 0$ , we have for every polynomial  $p \in \mathbb{C}[X]$ :

$$\tau(p(h^*h)) = \int_{-(q+1)}^{q+1} p(t^2) d\mu_{\tilde{h}}(t).$$

Hence, by Theorem 6.2(iv) and Eq. (47)

$$(60) \quad \eta_n = \tau(h_{2n}) = \int_{-(q+1)}^{q+1} Q_n^{(1)}(t^2) d\mu_{\tilde{h}}(t) = \int_{-(q+1)}^{q+1} Q_{2n}(t) d\mu_{\tilde{h}}(t), \quad n \geq 1.$$

By Proposition 6.1(i) we get (i), and by Proposition 6.1(ii) and Proposition 6.8(iv) we get (ii).  $\square$

Define

$$(61) \quad m_n^{(q)} := \binom{2n}{n} q^n - (q-1) \sum_{k=0}^{n-1} \binom{2n}{k} q^k.$$

Later on, we will show that these are the even moments for the measure in (63).

**Proposition 6.10.** *For  $n \geq 1$ , we have*

$$(62) \quad m_n = m_n^{(q)} + \sum_{k=0}^{n-1} \binom{2n}{k} q^k \zeta_{n-k}.$$

*Proof.* By Euler's Formula and the binomial theorem, we have

$$\cos^{2n} \theta = \frac{1}{2^{2n}} \left( \binom{2n}{n} + 2 \sum_{k=0}^{n-1} \binom{2n}{k} \cos 2(n-k)\theta \right).$$

Substituting  $t = \cos \theta$ , we get

$$t^{2n} = \frac{1}{2^{2n}} \left( \binom{2n}{n} + 2 \sum_{k=0}^{n-1} \binom{2n}{k} T_{2n-2k}(t) \right),$$



for  $-1 \leq t \leq 1$ , and hence also for all  $t \in \mathbb{R}$  as both sides of the equality are polynomials. Substituting  $t$  with  $\frac{t}{2\sqrt{q}}$  we get

$$t^{2n} = \binom{2n}{n} q^n + 2 \sum_{k=0}^{n-1} \binom{2n}{k} T_{2n-2k} \left( \frac{t}{2\sqrt{q}} \right) q^n,$$

Integrating both sides with respect to  $\mu_{\tilde{h}}$  we get by Eq. (59) and Proposition 6.9(ii) that

$$m_n = \binom{2n}{n} q^n + \sum_{k=0}^{n-1} \binom{2n}{k} (\zeta_{n-k} - q + 1) q^k,$$

which proves the proposition.  $\square$

A simple reformulation of (62) yields a formula for computing the cyclic numbers  $\zeta_n$  from the moments  $m_n$ :

**Corollary 6.11.** *We have  $\zeta_1 = m_1 - m_1^{(q)}$ , and*

$$\zeta_n = m_n - m_n^{(q)} - \sum_{k=1}^{n-1} \binom{2n}{k} q^k \zeta_{n-k} \quad (n > 1).$$

## 7. AMENABILITY, LEINERT SETS AND COGROWTH

### 7.1. Leinert sets.

**Definition 7.1** ([26], Definition III.B in [1]). A subset  $Y$  of a group  $\Gamma$  is called a Leinert set if for all  $n \in \mathbb{N}$ , all tuples  $(s_1, \dots, s_{2n}) \in E_{2n}$  satisfy

$$s_1^{-1} s_2 s_2^{-1} \cdots s_{2n-1}^{-1} s_{2n} \neq e.$$

By Theorem IIIF and Theorem IID(b) in [1] we have

- (i) If  $Y \subset \Gamma$  and  $Y \cap Y^{-1} = \emptyset$  then  $Y \cup Y^{-1}$  is a Leinert set if and only if  $Y$  generates freely a copy of the free group  $\mathbb{F}_{|Y|}$  with  $|Y|$  generators inside  $\Gamma$ .
- (ii) If  $Y \subset \Gamma$  and  $e \notin Y$  then  $Y \cup \{e\}$  is a Leinert set if and only if  $Y$  generates freely a copy of  $\mathbb{F}_{|Y|}$  inside  $\Gamma$ .

By the following theorem, due to Kesten and Lehnert, the norm of  $h$  from Section 6 is bounded by two values, the lower bound is related to Leinert sets and the upper bound to amenability.

**Theorem 7.2** ([23], [25]). *Let  $Y$  be a finite set in a discrete group  $\Gamma$  with  $|Y| = q+1$  elements ( $q \geq 2$ ). Let  $h := \sum_{s \in Y} s$ . Then*

- (i)  $2\sqrt{q} \leq \|h\| \leq q+1$ .
- (ii)  $\|h\| = q+1$  if and only if the subgroup  $\Gamma_0 := \langle Y^{-1}Y \rangle \subset \Gamma$  generated by  $Y^{-1}Y$  is amenable.
- (iii)  $\|h\| = 2\sqrt{q}$  if and only if  $Y$  is a Leinert set.

*Proof.* (i): The upper bound is trivial since each  $s$  is a unitary operator in  $L(\Gamma)$ . The proof of the lower bound is the following: Write  $Y = \{s_1, \dots, s_{q+1}\}$ . Apply now Proposition 2 and Proposition 5 of [25] to  $G = \mathbb{F}_{q+1}$ ,  $H = \Gamma$ , and  $\rho : G \rightarrow H$  the unique group homomorphism for which

$$\rho(t_i) = s_i, \quad (1 \leq i \leq q+1),$$

where  $t_1, \dots, t_{q+1}$  are the generators of  $\mathbb{F}_{q+1}$ . Then

$$\left\| \sum_{s \in Y} s \right\|_{L(\Gamma)} \geq \left\| \sum_{i=1}^{q+1} t_i \right\|_{L(\mathbb{F}_{q+1})} = 2\sqrt{q}.$$

(ii): We have

$$h^*h = \sum_{s_1, s_2 \in Y} s_1^{-1} s_2 = \sum_{s \in Y^{-1}Y} c_s s,$$

where  $c_{s^{-1}} = \bar{c}_s = c_s$  for all  $s \in Y^{-1}Y$  because  $h^*h$  is self-adjoint. Moreover,  $Y^{-1}Y$  is a symmetric set (i.e.  $Y^{-1}Y = (Y^{-1}Y)^{-1}$ ), and

$$\sum_{s \in Y^{-1}Y} c_s = |Y|^2 = (q+1)^2.$$

Hence, by Section 3 of [23],  $\Gamma_0$  is amenable if and only if  $\|h^*h\| = (q+1)^2$ . This proves (ii) because  $\|h^*h\| = \|h\|^2$ .

(iii): The proof of (iii) follows from Theorem 9 of [25].  $\square$

Note that in view of Remark 7.3 below, Theorem 1.2 and Theorem 1.3 in the introduction are both special cases of Theorem 7.2.

**Remark 7.3.** If  $Y$  is a symmetric set (i.e.  $Y = Y^{-1}$ ) it follows immediately from Kesten's Theorem, that  $\|h\| = q+1$  if and only if the subgroup  $\langle Y \rangle \subset \Gamma$  generated by  $Y$  is amenable. This follows also from (ii) because  $\langle Y^2 \rangle \subset \langle Y \rangle$  is a subgroup of  $\langle Y \rangle$  of index at most 2. (Recall that, amenability of a group is always preserved by any subgroup. On the other hand, amenability of a subgroup is preserved by the group if the index is finite.)

**Theorem 7.4.** *Let  $Y \subset \Gamma$  be a finite subset in a discrete group  $\Gamma$  with  $|Y| = q+1$  elements, where  $q \geq 1$ . Then, with the notation from Section 6, the following are equivalent:*

- (i)  $Y$  is a Leinert set.
- (ii)  $\|h_n\|_2^2 = (q+1)q^{n-1}$  for all  $n \in \mathbb{N}$ .
- (iii)  $\xi_n = 0$  for all  $n \in \mathbb{N}$ .
- (iv)  $\eta_n = 0$  for all  $n \in \mathbb{N}$ .
- (v)  $\zeta_n = 0$  for all  $n \in \mathbb{N}$ .
- (vi)  $m_n = m_n^{(q)}$ , where  $m_n^{(q)}$  is given by Eq. (61).
- (vii)  $\mu_{\tilde{h}} = \mu^{(q)}$  where

$$(63) \quad \mu^{(q)} = \frac{q+1}{2\pi} \frac{(4q-t^2)^{1/2}}{(q+1)^2 - t^2} 1_{[-2\sqrt{q}, 2\sqrt{q}]}(t) dt.$$

*Proof.* (i)  $\iff$  (iv): Recall that

$$\eta_n = \tau(h_{2n}) = \tau\left(\sum_{(s_1, \dots, s_{2n}) \in E_{2n}} s_1^{-1} s_2 s_3^{-1} \cdots s_{2n-1} s_{2n}\right).$$

Since  $\tau(g) = \delta_{g,e}$  for  $g \in \Gamma$  and a Leinert set omits the identity the equivalence follows.

- (ii)  $\iff$  (iii) This follows by definition of  $\xi_n$ .
- (iii)  $\iff$  (iv)  $\iff$  (v): This follows from Proposition 6.8.
- (v)  $\iff$  (vi): This follows from Proposition 6.10.

(iv)  $\iff$  (vii): By (60) we have for  $n \in \mathbb{N}$  that

$$\eta_n = \tau(h_{2n}) = \int_I Q_{2n}(t) d\mu_{\tilde{h}}(t)$$

where  $I = [-(q+1), q+1]$ . Moreover, since  $\mu_{\tilde{h}}$  is a symmetric measure (cf. Section 2), and  $Q_m$  is an odd polynomial when  $m$  is odd,

$$\int_I Q_m(t) d\mu_{\tilde{h}}(t) = 0 \quad m = 1, 3, 5, \dots$$

Hence (iv) is equivalent to

$$(64) \quad \int_I Q_m(t) d\mu_{\tilde{h}}(t) = 0, \quad m \in \mathbb{N}.$$

Since  $\mu_{\tilde{h}}$  is also a probability measure

$$(65) \quad \int_I 1 d\mu_{\tilde{h}}(t) = 1.$$

Since  $\text{span}\{1, Q, Q_2, \dots\}$  is the set of all polynomials in  $\mathbb{C}[X]$  it follows from Weierstrass' approximation theorem and Riesz Representation theorem that  $m_{\tilde{h}}$  is uniquely determined by (64) and (65). In [32] pp. 283-284 (see also [9]) a sequence of polynomials  $(p_n)_{n=0}^\infty$  is defined by

$$\begin{aligned} p_0(t) &= 1 \\ p_1(t) &= t \\ p_2(t) &= t^2 - (a+1) \\ p_{n+1}(t) &= tp_n(t) - ap_n(t) \end{aligned}$$

for a fixed number  $a \in \mathbb{N}$ , and it is proven in [32] p. 284 that

$$(66) \quad \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} p_k(t) p_\ell(t) \frac{(4a - t^2)^{1/2}}{(a+1)^2 - t^2} dt = \frac{a^{k-1}}{\lambda_k} \delta_{k\ell},$$

where  $\lambda_k = 1$  for  $k \in \mathbb{N}$  and  $\lambda_0 = \frac{a+1}{a}$ . Letting now  $a = q$ , then for  $n \geq 1$ ,  $p_n(t)$  coincide with our polynomials  $Q_n(t)$ . Hence using (66) first with  $k \in \mathbb{N}$  and  $\ell = 0$ , and next with  $k = \ell = 0$  we get

$$(67) \quad \frac{1}{2\pi} \int_{-2\sqrt{q}}^{2\sqrt{q}} Q_n(t) \frac{(4q - t^2)^{1/2}}{(q+1)^2 - t^2} dt = 0 \quad n \in \mathbb{N}$$

and

$$(68) \quad \frac{1}{2\pi} \int_{-2\sqrt{q}}^{2\sqrt{q}} \frac{(4q - t^2)^{1/2}}{(q+1)^2 - t^2} dt = \frac{1}{q+1} \quad n \in \mathbb{N}.$$

Hence, the measure  $\mu^{(q)}$  defined in (63) satisfies (64) and (65). Therefore (iv) is equivalent to that  $\mu_{\tilde{h}} = \mu^{(q)}$ .  $\square$

**Corollary 7.5.** *The odd moments of  $\mu^{(q)}$  are zero. The even moments of  $\mu^{(q)}$  are given by  $(m_n^{(q)})_{n=1}^\infty$ , i.e.*

$$\int_I t^{2n} d\mu^{(q)}(t) dt = m_n^{(q)}, \quad n \in \mathbb{N},$$

where

$$m_n^{(q)} = \binom{2n}{n} q^n - (q-1) \sum_{k=0}^{n-1} \binom{2n}{k} q^k.$$

*Proof.* Let  $Y$  be a Leinert set with  $|Y| = q+1$  elements. By Theorem 7.4,  $\mu_{\tilde{h}} = \mu^{(q)}$  and  $m_n = m_n^{(q)}$ . Hence

$$\int_I t^{2n} d\mu^{(q)}(t) dt = \int_I t^{2n} d\mu_{\tilde{h}}(t) dt = m_n = m_n^{(q)}.$$

□

## 7.2. Connection to the cogrowth coefficients of Cohen and Grigorchuk.

Let  $X$  be a finite set of generators of a group  $\Gamma$  such that  $X \cap X^{-1} = \emptyset$  and  $|X| \geq 2$ . Let  $Y := X \cup X^{-1}$  and  $q = |Y| - 1 = 2|X| - 1$ . In [11] and [16], Cohen and Grigorchuk independently introduced the notion of cogrowth coefficients  $(\gamma_n)_{n=1}^\infty$  for  $(\Gamma, X)$ , by putting  $\gamma_n$  equal to the number of elements in the set

$$\{(s_1, \dots, s_n) \in Y^n \mid s_{i+1} \neq s_i^{-1} (1 \leq i \leq n-1) \text{ and } s_1 s_2 \dots s_n = e\}$$

As Cohen puts it,  $\gamma_n$  is the number of reduced words in  $Y$  of length  $n$ , which represent the unit element of  $\Gamma$ . Since  $\tau(g) = \delta_{g,e}$ ,  $g \in \Gamma$  we have

$$(69) \quad \gamma_n = \sum_{(s_1, \dots, s_n) \in E_n} \tau(s_1 s_2 \dots s_n).$$

Note that since  $Y = Y^{-1}$ , we have  $\gamma_{2n} = \eta_n$  according to the definition of the reduced numbers  $\eta_n$  in (56). Cohen proved in pp. 302-303 in [11] that if  $|X| \geq 2$  and  $X$  does not generate  $\Gamma$  freely, then

$$(70) \quad \gamma = \lim_{n \rightarrow \infty} \gamma_{2n}^{\frac{1}{2n}} (= \lim_{n \rightarrow \infty} \eta_n^{\frac{1}{2n}})$$

exists and  $\gamma \in (\sqrt{q}, q]$ . Moreover if we let  $h = \sum_{s \in Y} s$ , (assuming still that  $X$  does not generate  $\Gamma$  freely) by Theorem 3 in [11]:

$$(71) \quad \gamma + \frac{q}{\gamma} = \|h\|.$$

Since  $\gamma > \sqrt{q}$ , it follows that

$$(72) \quad \gamma = \frac{1}{2}(\|h\| + \sqrt{\|h\|^2 - 4q}).$$

We next prove the following extension of the above:

**Theorem 7.6.** *Let  $Y \subset \Gamma$  be a finite set with  $|Y| = q+1$  elements. Let  $h = \sum_{s \in Y} s$  as in Theorem 7.2, and let  $(\xi_n)$ ,  $(\eta_n)$   $(\zeta_n)$  be as in (56) and let*

$$\gamma := \frac{1}{2}(\|h\| + (\|h\|^2 - 4q)^{1/2}).$$

Then

$$(73) \quad \sqrt{q} \leq \gamma \leq q \quad \text{and} \quad \gamma + \frac{q}{\gamma} = \|h\|.$$

Moreover, if  $Y$  is not a Leinert set, and  $q \geq 2$  then  $\gamma > \sqrt{q}$  and

$$(74) \quad \gamma = \lim_{n \rightarrow \infty} \|h_n\|_2^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \xi_n^{\frac{1}{2n}} = \lim_{n \rightarrow \infty} \eta_n^{\frac{1}{2n}} = \lim_{n \rightarrow \infty} \zeta_n^{\frac{1}{2n}}.$$

*Proof.* By Theorem 7.2

$$2\sqrt{q} \leq \|h\| \leq q + 1.$$

Hence  $\|h\|^2 - 4q \geq 0$ . Let

$$(75) \quad \gamma = \frac{1}{2}(\|h\| + (\|h\|^2 - 4q)^{1/2})$$

$$(76) \quad \gamma' = \frac{1}{2}(\|h\| - (\|h\|^2 - 4q)^{1/2}).$$

Then

$$\gamma + \gamma' = \|h\| \quad \gamma\gamma' = q \quad \gamma \geq \gamma' \geq 0$$

Hence

$$(77) \quad \gamma \geq \sqrt{q} \quad \text{and} \quad \gamma + \frac{q}{\gamma} = \gamma + \gamma' = \|h\|.$$

The function

$$f(t) = t + \frac{q}{t}$$

is strictly increasing on the interval  $[\sqrt{q}, \infty)$ , and  $f(\gamma) = \|h\| \leq q + 1 = f(q)$ . Hence  $\gamma \leq q$ , which proves (73). Assume next that  $q \geq 2$  and  $Y$  is not a Leinert set. Then by Theorem 7.2,  $\|h\|^2 - 4q > 0$ . Hence  $\gamma > \gamma'$ , which shows that  $\gamma > \sqrt{q}$ . We next prove (74). Since  $\|h_n\|_2^2 \geq \xi_n \geq \eta_n \geq \zeta_n \geq 0$  (Proposition 6.6), it is sufficient to show that

$$(78) \quad \limsup_{n \rightarrow \infty} \|h_n\|_2^{\frac{1}{n}} \leq \gamma$$

$$(79) \quad \liminf_{n \rightarrow \infty} \zeta_n^{\frac{1}{2n}} \geq \gamma.$$

Note that (78) follows immediately from the following lemma:

**Lemma 7.7.** *For all  $n \geq \mathbb{N}$ ,*

$$\zeta_n \leq 3\gamma^{2n}, \quad \eta_n \leq 3n\gamma^{2n}, \quad \xi_n \leq 3n^2\gamma^{2n}, \quad \|h_n\|_2^2 \leq 5n^2\gamma^{2n}.$$

*Proof.* By Proposition 6.9 we have

$$(80) \quad \zeta_n = (q - 1) + 2q^n \int_{-(q+1)}^{q+1} T_{2n} \left( \frac{t}{2\sqrt{q}} \right) d\mu_{\tilde{h}}(t)$$

where  $\mu_{\tilde{h}}$  is the spectral distribution of  $\tilde{h} = \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix}$ . In particular,

$$(81) \quad \text{supp}(\mu_{\tilde{h}}) \subset [-\|h\|, \|h\|].$$

By formula (2) in page 184 of [20]

$$\begin{aligned} T_k(\cos \theta) &= \cos(k\theta), & \theta &\in [0, \pi] \\ T_k(\cosh u) &= \cosh(ku), & u &\geq 0. \end{aligned}$$

Hence,  $|T_k(t)| \leq 1$  for  $t \in [-1, 1]$ ,  $T_k(1) = 1$ , and  $T_k$  is strictly increasing on  $[1, \infty)$ . Since also  $T_k(-x) = (-1)^k T_k(x)$ , we have

$$\max_{|t| \leq t_0} |T_k(t)| = T_k(t_0), \quad \forall t_0 \geq 1.$$

Hence by (80) and (81)

$$(82) \quad \zeta_n \leq q - 1 + 2q^n T_{2n} \left( \frac{\|h\|}{2\sqrt{q}} \right).$$

By (77)  $t := \log \frac{\gamma}{\sqrt{q}} \geq 0$  is non negative. Then

$$\cosh t = \frac{1}{2} \left( \frac{\gamma}{\sqrt{q}} + \frac{\sqrt{q}}{\gamma} \right) = \frac{\|h\|}{2\sqrt{q}}$$

which implies that

$$T_{2n} \left( \frac{\|h\|}{2\sqrt{q}} \right) = \cosh(2nt) = \frac{1}{2} \left( \left( \frac{\gamma}{\sqrt{q}} \right)^{2n} + \left( \frac{\sqrt{q}}{\gamma} \right)^{2n} \right) \leq \left( \frac{\gamma}{\sqrt{q}} \right)^{2n}.$$

This proves that

$$\zeta_n \leq q - 1 + 2\gamma^{2n} \leq 3\gamma^{2n}, \quad n \in \mathbb{N}$$

where the last inequality follows from the inequality  $\gamma \geq \sqrt{q}$ . From Proposition 6.8(iii) we get, using again  $\gamma \geq \sqrt{q}$  that

$$\begin{aligned} \eta_n &\leq \zeta_n + q\zeta_{n-1} + q^2\zeta_{n-2} + \dots + q^{n-1}\zeta_1 \\ &\leq 3\gamma^{2n} \left( 1 + \frac{q}{\gamma^2} + \dots + \left( \frac{q}{\gamma^2} \right)^{n-1} \right) \\ &\leq 3n\gamma^{2n}. \end{aligned}$$

In particular,  $\eta_k \leq 3n\gamma^{2k}$ ,  $1 \leq k \leq n$ . Applying Proposition 6.8(i) to this inequality we get in the same way that  $\xi_n \leq 3n^2\gamma^{2n}$ . Hence

$$\|h_n\|_2^2 = \xi_n + (q+1)q^{n-1} \leq \xi_n + 2q^n \leq 3n^2\gamma^{2n} + 2\gamma^{2n} \leq 5n^2\gamma^{2n}.$$

□

*Proof of Theorem 7.6 (continued).* We prove now (79). Recall that  $\gamma > \sqrt{q}$ , and let  $\gamma_1 \in (\sqrt{q}, \gamma)$  be arbitrary. Let

$$\alpha := \gamma_1 + \frac{q}{\gamma_1}.$$

Since  $f(\gamma) := \gamma + \frac{q}{\gamma}$  is strictly increasing on  $[\sqrt{q}, \infty)$ , we have

$$(83) \quad 2\sqrt{q} < \alpha < \|h\|.$$

Since, by (13),  $\pm\|h\| \in \text{supp}(\mu_{\tilde{h}})$  we also have  $\mu_{\tilde{h}}([\alpha, \|h\|]) > 0$ . We have previously seen that  $|T_{2n}(t)| \leq 1$  for  $t \in [-1, 1]$  and that  $T_{2n}$  is positive and increasing on  $[1, \infty)$ . Hence

$$\begin{aligned} \zeta_n &= (q-1) + 2q^n \int_{-(q+1)}^{q+1} T_{2n} \left( \frac{t}{2\sqrt{q}} \right) d\mu_{\tilde{h}}(t) \\ &\geq (q-1) + 2q^n \left( \int_{-2\sqrt{q}}^{2\sqrt{q}} (-1) d\mu_{\tilde{h}}(t) + \int_{\alpha}^{\|h\|} T_{2n} \left( \frac{t}{2\sqrt{q}} \right) d\mu_{\tilde{h}}(t) \right) \\ &\geq q-1 - 2q^n + 2q^n T_{2n} \left( \frac{\alpha}{2\sqrt{q}} \right) \mu_{\tilde{h}}([\alpha, \|h\|]). \end{aligned}$$

Let now  $u := \log \frac{\gamma_1}{\sqrt{q}} > 0$ . Then

$$\cosh u = \frac{1}{2} \left( \frac{\gamma_1}{\sqrt{q}} + \frac{\sqrt{q}}{\gamma_1} \right) = \frac{\alpha}{2\sqrt{q}}.$$

Hence

$$T_{2n} \left( \frac{\alpha}{2\sqrt{q}} \right) = \cosh(2nu) = \frac{1}{2} \left( \left( \frac{\gamma_1}{\sqrt{q}} \right)^{2n} + \left( \frac{\sqrt{q}}{\gamma_1} \right)^{2n} \right) \geq \frac{1}{2} \frac{\gamma_1^{2n}}{q^n}$$

which shows that

$$\zeta_n \geq q - 1 - 2q^n + \gamma_1^{2n} \mu_{\tilde{h}}([\alpha, \|h\|]).$$

Since  $\gamma_1 > \sqrt{q}$  it follows that

$$\liminf_{n \rightarrow \infty} \zeta_n^{1/2n} \geq \gamma_1$$

and since  $\gamma_1 \in (\sqrt{q}, \gamma)$  was arbitrary we have

$$\liminf_{n \rightarrow \infty} \zeta_n^{1/2n} \geq \gamma$$

proving (79).  $\square$

**Remark 7.8.** If  $Y$  is a Leinert set, then  $\gamma = \sqrt{q}$  by Theorem 7.2(iii). Hence by Theorem 7.4

$$\lim_{n \rightarrow \infty} \|h_n\|_2^{\frac{1}{n}} = \sqrt{q} = \gamma$$

while

$$\lim_{n \rightarrow \infty} \xi_n^{\frac{1}{2n}} = \lim_{n \rightarrow \infty} \eta_n^{\frac{1}{2n}} = \lim_{n \rightarrow \infty} \zeta_n^{\frac{1}{2n}} = 0.$$

**Corollary 7.9.** *Let  $\Gamma$  be a discrete group and let  $Y \subset \Gamma$  be a finite set with  $|Y| = q + 1$  elements ( $q \geq 2$ ), such that  $Y^{-1}Y$  generates  $\Gamma$ . Then the following are equivalent:*

- (i)  $\Gamma$  is amenable
- (ii)  $\lim_{n \rightarrow \infty} \|h_n\|_2^{1/n} = q$
- (iii)  $\lim_{n \rightarrow \infty} \xi_n^{\frac{1}{2n}} = q$
- (iv)  $\lim_{n \rightarrow \infty} \eta_n^{\frac{1}{2n}} = q$
- (v)  $\lim_{n \rightarrow \infty} \zeta_n^{\frac{1}{2n}} = q$ .

Note as well that  $\Gamma$  is amenable if and only if  $\lim_{n \rightarrow \infty} m_n^{\frac{1}{2n}} = q + 1$ .

*Proof.* If  $\Gamma$  is amenable, then by Theorem 7.2(ii),  $\|h\| = q + 1$  and thus  $\gamma = \frac{1}{2}(\|h\| + \sqrt{\|h\|^2 - 4q}) = q$ . Since  $q \geq 2$ ,  $\|h\| = q + 1 > 2\sqrt{q}$ . Hence by Theorem 7.2(iii)  $Y$  is not a Leinert set, and (ii), (iii), (iv), (v) in the corollary follows now from formula (74) in Theorem 7.6. Conversely, if one of the statements (ii), (iii), (iv), or (v) in the corollary holds, then by Remark 7.8,  $Y$  is not a Leinert set, so using again formula (74) in Theorem 7.6 we have  $\gamma = q$ , and hence by (73),  $\|h\| = \gamma + \frac{q}{\gamma} = q + 1$ , which by Theorem 7.2(ii) implies that  $\Gamma$  is amenable. By Proposition 4.1

$$\lim_{n \rightarrow \infty} m_n^{\frac{1}{2n}} = \|h\|,$$

which proves the last statement in the corollary.  $\square$

## APPENDIX A. ON CONNECTEDNESS OF SPECTRA OF ELEMENTS IN $C_r^*(F)$

In [15] Farley proved that the Thompson group  $F$  admits a proper affine isometric action on a Hilbert space or equivalently,  $F$  has the Haagerup property (cf. [10]). Hence by the result of Higson and Kasparov [21],  $F$  satisfies the Baum-Connes conjecture with coefficients, which in turn implies that  $F$  satisfies the Kadison-Kaplansky conjecture, i.e. (since  $F$  is torsion free) the reduced  $C^*$ -algebra  $C_r^*(F)$

has no projections other than 0 or 1 (see e.g. [30]). Here projections mean self-adjoint idempotents ( $p^2 = p = p^*$ ). This implies that the spectrum of every self-adjoint operator  $a \in C_r^*(F)$  is a connected subset of  $\mathbb{R}$ . The following two propositions are simple applications of this:

**Proposition A.1** (Case 1). *Let  $h = I + A + B$  and  $\tilde{h} = \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix}$ . Then there exists a  $\delta \in [0, \|h\|)$  such that*

$$\sigma(\tilde{h}) = \text{supp}(\mu_{\tilde{h}}) = [-\|h\|, -\delta] \cup [\delta, \|h\|].$$

*Proof.* Since  $h^*h$  is a self-adjoint element of  $C_r^*(F)$ , its spectrum is a connected subset of  $\mathbb{R}$ . Hence  $\sigma(h^*h) = [\alpha, \beta]$ , where  $\alpha = \min(\sigma(h^*h))$  and  $\beta = \max(\sigma(h^*h))$ . Moreover, since  $h^*h \geq 0$ ,

$$0 \leq \alpha \leq \beta \quad \text{and} \quad \beta = \|h^*h\| = \|h\|^2.$$

Since  $\mu_{h^*h}$  is the image measure of the symmetric measure  $\mu_{\tilde{h}}$  by the map  $t \rightarrow t^2$  (see the end of Section 2), we have

$$\sigma(\tilde{h}) = \text{supp}(\mu_{\tilde{h}}) = [-\|h\|, -\sqrt{\alpha}] \cup [\sqrt{\alpha}, \|h\|].$$

Put  $\delta = \sqrt{\alpha}$ . Then  $\delta \in [0, \|h\|]$ . To see that  $\delta < \|h\|$ , note first that

$$\tau(h^*h) = 3$$

because  $\{I, A, B\}$  is an orthonormal set with respect to the inner product  $\langle a, b \rangle = \tau(b^*a)$ . Hence

$$3 = \tau(h^*h) = \int_{\alpha}^{\beta} t d\mu_{h^*h}(t) \geq \int_{\alpha}^{\beta} \alpha d\mu_{h^*h}(t) = \alpha.$$

Hence  $\delta = \sqrt{\alpha} \leq \sqrt{3} < 2\sqrt{2} \leq \|h\|$  by Corollary 1.4.  $\square$

**Proposition A.2** (Case 2). *Let  $h = A + A^{-1} + B + B^{-1}$  and  $\tilde{h} = \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix}$ .*

Then

$$\sigma(h) = \text{supp}(\mu_h) = \text{supp}(\mu_{\tilde{h}}) = [-\|h\|, \|h\|].$$

*Proof.* We know that  $\mu_h = \mu_{\tilde{h}}$  (see end of Section 5). Hence

$$\sigma(h) = \text{supp}(\mu_h) = \text{supp}(\mu_{\tilde{h}})$$

and from Section 2, we have

$$\text{supp}(\mu_{\tilde{h}}) \subset [-\|h\|, \|h\|]$$

and

$$\pm\|h\| \in \text{supp}(\mu_{\tilde{h}}),$$

but since  $h = h^* \in C_r^*(F)$ ,  $\sigma(h)$  is a connected subset of  $\mathbb{R}$ . Hence

$$\text{supp}(\mu_{\tilde{h}}) = \sigma(h) = [-\|h\|, \|h\|],$$

proving the proposition.  $\square$

**Remark A.3.** By some extra work, it can be proved also for Case 1 that

$$\text{supp}(\mu_{\tilde{h}}) = [-\|h\|, \|h\|].$$

However, in the rest of this section, we shall only need the following immediate corollary of Proposition A.1 and Proposition A.2:



**Corollary A.4.** *In both Case 1 and Case 2,  $\sigma(\tilde{h}) = \text{supp}(\mu_{\tilde{h}})$  is an infinite subset of  $\mathbb{R}$  without isolated points.*

Let  $H$  be a Hilbert space, and let  $K(H) \subset B(H)$  be the set of compact operators on  $H$  and let  $\rho$  denote the quotient map  $\rho : B(H) \rightarrow B(H)/K(H)$ . Then the essential spectrum  $\sigma_{\text{ess}}(T)$  of an element  $T \in B(H)$  is the spectrum  $\sigma(\rho(T))$  of  $\rho(T)$ , see e.g. p. 30 in [28]. Note that  $\sigma_{\text{ess}}(T) \subset \sigma(T)$  and for every  $K \in K(H)$ ,  $\sigma_{\text{ess}}(T + K) = \sigma_{\text{ess}}(T)$ . For a self-adjoint operator  $S = S^* \in B(H)$ , it is easy to see that  $\lambda \in \sigma_{\text{ess}}(S)$  if and only if for every  $\varepsilon > 0$ , the spectral projection  $1_{(\lambda-\varepsilon, \lambda+\varepsilon)}(S)$  of  $S$  is infinite dimensional. The following result is well known (cf. Theorem VII.10 in [31]).

**Proposition A.5.** *Let  $S = S^* \in B(H)$ . Then  $\lambda \in \sigma(S) \setminus \sigma_{\text{ess}}(S)$  if and only if the following two conditions hold*

- (i)  $\lambda$  is an isolated point in  $\sigma(S)$ .
- (ii)  $\lambda$  is an eigenvalue of  $S$  and  $\dim(\ker(S - \lambda I)) < \infty$ .

**Corollary A.6.** *If  $S = S^* \in B(H)$  and  $\sigma(S)$  has no isolated points then  $\sigma_{\text{ess}}(S) = \sigma(S)$ .*

**Proposition A.7.** *Let  $M \in B(\ell^2(\mathbb{N}_0))$  be an operator of the form*

$$(84) \quad M = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & \cdots \\ \alpha_1 & 0 & \alpha_2 & 0 & \cdots \\ 0 & \alpha_2 & 0 & \alpha_3 & \cdots \\ 0 & 0 & \alpha_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where  $\alpha_n \geq 0$  for all  $n \in \mathbb{N}$  and assume that  $\sigma(M)$  has no isolated points. Then

$$(85) \quad \liminf_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n) \leq \|M\| \leq \limsup_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n).$$

*Proof.* The inequality on the left hand side follows from the proof of Proposition 4.4 (without any assumption on  $\sigma(M)$ ). Put

$$(86) \quad N_n = \begin{pmatrix} 0 & \alpha_{n+1} & 0 & 0 & \cdots \\ \alpha_{n+1} & 0 & \alpha_{n+2} & 0 & \cdots \\ 0 & \alpha_{n+2} & 0 & \alpha_{n+3} & \cdots \\ 0 & 0 & \alpha_{n+3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad n \in \mathbb{N}.$$

Then  $\|N_n\| = \|Q_n M Q_n\| \leq \|M\|$ , where  $Q_n$  denotes the projection on  $\text{span}\{\delta_k \mid k \geq n\}$ , and  $(\delta_k)_{k=0}^\infty$  is the standard basis for  $\ell^2(\mathbb{N}_0)$ . But since  $Q_n M Q_n$  is a compact perturbation of  $M$ ,  $\sigma_{\text{ess}}(Q_n M Q_n) = \sigma_{\text{ess}}(M)$  so by Corollary A.6

$$\sigma(Q_n M Q_n) \supset \sigma_{\text{ess}}(Q_n M Q_n) = \sigma_{\text{ess}}(M) = \sigma(M),$$

and since the norm of a self-adjoint operator is equal to its spectral radius, also

$$\|Q_n M Q_n\| \geq \|M\|, \quad n \in \mathbb{N},$$

so altogether,

$$\|N_n\| = \|Q_n M Q_n\| = \|M\|, \quad n \in \mathbb{N}.$$

By the proof of Proposition 4.4,

$$\|M\| = \|N_n\| \leq \sup\{\alpha_{k-1} + \alpha_k \mid k \geq n+2\}.$$

Hence

$$\|M\| \leq \limsup_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n).$$

□

**Corollary A.8.** *Let  $h = I + A + B$  (case 1) or  $h = A + A^{-1} + B + B^{-1}$  (case 2) and let*

$$(87) \quad M = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & \cdots \\ \alpha_1 & 0 & \alpha_2 & 0 & \cdots \\ 0 & \alpha_2 & 0 & \alpha_3 & \cdots \\ 0 & 0 & \alpha_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

be the operator in  $B(\ell^2(\mathbb{N}_0))$  built from  $\mu_{\tilde{h}}$  as in Section 4, then

$$(88) \quad \liminf_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n) \leq \|M\| \leq \limsup_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n).$$

*Proof.* By the proof of Proposition 4.3,  $M$  is unitarily equivalent to the multiplication operator  $m_t$  on  $L^2(\mu_{\tilde{h}})$  given by  $f(t) \mapsto tf(t)$ ,  $f \in L^2(\mu_{\tilde{h}})$ . Hence  $\sigma(M) = \sigma(m_t) = \text{supp}(\mu_{\tilde{h}}) = \sigma(\tilde{h})$ . Formula (88) now follows from Proposition A.7 and Corollary A.4. □

## APPENDIX B. A NUMBER THEORETICAL TEST

This section contains a proof of the test for computational errors mentioned at the end of Section 3. Let  $\Gamma$  be a group and  $Y \subset \Gamma$  a finite subset of  $\Gamma$  with  $|Y| = q + 1$  elements ( $q \geq 2$ ). Moreover, let  $m_n, \|h_n\|_2^2, \xi_n, \eta_n, \zeta_n$  be the sequences of numbers introduced in Sections 3 and 6. Then the following holds:

- Theorem B.1.** (i)  $\xi_1 = \eta_1 = \zeta_1 = 0$  and  $m_1 = \|h_1\|_2^2 = q + 1$ .  
(ii) For  $n \geq 2$  the numbers  $\xi_n, \eta_n, \zeta_n, \|h_n\|_2^2$  are all even integers, while the  $m_n$  numbers have the same parity as the number  $q + 1$ .  
(iii) Let  $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$  denote the Möbius function, i.e.  $\mu(1) := 1$  and for  $n \geq 2$ ,
- $$\mu(n) := \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ (product of } k \text{ distinct prime numbers),} \\ 0 & \text{otherwise.} \end{cases}$$

If  $\Gamma$  is torsion free, then the numbers

$$(89) \quad \zeta'_n := \sum_{d|n} \mu\left(\frac{n}{d}\right) \zeta_d, \quad n \in \mathbb{N}$$

are non-negative integers divisible by  $2n$ . In particular, if  $n = p$  is a prime number, then  $\zeta_p = \zeta'_p$  and hence  $\zeta_p$  is divisible by  $2p$ .

*Proof.* (i): By definition

$$\eta_1 = |\{(s_1, s_2) \in Y^2 \mid s_1 \neq s_2, s_1^{-1} s_2 = e\}| = 0.$$

Hence, by Propositions 6.8 and (56),  $\xi_1 = \zeta_1 = 0$  and  $m_1 = \|h_1\|_2^2 = q + 1$ .

(ii): Recall that for  $n \in \mathbb{N}$ :

$$\tilde{E}_{2n} = \{(s_1, \dots, s_{2n}) \in Y^{2n} \mid s_1 \neq s_2 \neq \dots \neq s_{2n} \neq s_1\}.$$

and  $\zeta_n = |R_n|$  where

$$R_n := \{(s_1, \dots, s_{2n}) \in \tilde{E}_{2n} \mid s_1^{-1} s_2 \cdots s_{2n-1}^{-1} s_{2n} = e\}.$$

Define the reversing map  $\sigma : \tilde{E}_{2n} \rightarrow \tilde{E}_{2n}$  by

$$\sigma(s_1, \dots, s_{2n}) := (s_{2n}, \dots, s_1).$$

Then  $\sigma^2 = id$  and  $\sigma(R_n) = R_n$ . Moreover, for each  $s = (s_1, \dots, s_{2n}) \in \tilde{E}_{2n}$ ,  $\sigma(s) \neq s$ , because  $s_1 \neq s_{2n}$ . Hence, all the orbits in  $\tilde{E}_{2n}$  under the action of the group  $\{id, \sigma\} \cong \mathbb{Z}_2$  have size 2. Since  $\sigma(R_n) = R_n$ ,  $\zeta_n = |R_n|$  is an even number. Thus, by Proposition 6.8,  $\eta_n$  and  $\xi_n$  are also even. Hence for  $n \geq 2$ ,

$$\|h_n\|_2^2 = \xi_n + (q+1)q^{n-1}$$

is also even. The last statement in (ii) follows from (61) and (62).

(iii): Let the reversing map  $\sigma : \tilde{E}_{2n} \rightarrow \tilde{E}_{2n}$  and  $R_n$  be as in the proof of (ii), and define  $\rho : \tilde{E}_{2n} \rightarrow \tilde{E}_{2n}$  by

$$\rho(s_1, \dots, s_{2n}) := (s_3, s_4, \dots, s_{2n}, s_1, s_2).$$

Then clearly  $\rho(R_n) = R_n$ ,  $\rho^n = id$ ,  $\sigma\rho\sigma = \rho^{-1}$ , and the group  $G$  of transformations of  $\tilde{E}_{2n}$  generated by  $\sigma$  and  $\rho$  is equal to

$$G = H \sqcup \sigma H$$

where  $H = \{id, \rho, \rho^2, \dots, \rho^{n-1}\} \cong \mathbb{Z}_n$ . Moreover,  $G \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$ , the dihedral group with  $2n$  elements. We prove next the following claim: For each  $s = (s_1, \dots, s_{2n}) \in \tilde{E}_{2n}$ , the stabilizer group

$$G_s = \{g \in G \mid gs = s\}$$

is contained in  $H$ .

To prove the claim, we just have to show that  $\sigma H \cap G_s = \emptyset$ . From the proof of (ii), we already know that  $\sigma \notin G_s$ . Let now  $k \in \{1, \dots, n-1\}$  and assume that  $\sigma\rho^k \in G_s$ . Then

$$(s_{2k}, s_{2k-1}, \dots, s_1, s_{2n}, s_{2n-1}, \dots, s_{2k+1}) = (s_1, \dots, s_{2n}).$$

By comparing the  $k$ 'th terms of the two tuples, we get  $s_{k+1} = s_k$ , which is a contradiction. Hence  $\sigma H \cap G_s = \emptyset$  proving the claim.

Since  $G_s$  is a subgroup of  $H \cong \mathbb{Z}_n$ ,  $G_s$  is of the form

$$H_d = \{\rho^k \mid k \text{ is a multiple of } d\},$$

where  $d \in \{1, \dots, n\}$  is a divisor of  $n$ . Note that  $|H_d| = \frac{n}{d}$ . In particular  $H_n = \{id\}$ . Assume now that  $\Gamma$  is torsion free and write

$$R_n = \bigsqcup_{d|n} R_n^d,$$

where  $R_n^d = \{s \in R_n \mid G_s = H_d\}$ . If  $s \in R_n^d$ , for a divisor  $d$  of  $n$ , for which  $d < n$ , then

$$(90) \quad s = (s_1, \dots, s_{2d}, s_1, \dots, s_{2d}, \dots, s_1, \dots, s_{2d})$$

where  $(s_1, \dots, s_{2d})$  is repeated  $\frac{n}{d}$  times, and  $\rho^k s \neq s$  for  $k = 1, \dots, d-1$ . Since  $s \in R_n$ ,

$$(s_1^{-1} s_2 \cdots s_{2d-1}^{-1} s_{2d})^{n/d} = e,$$

and since  $\Gamma$  is torsion free, also

$$s_1^{-1} s_2 \cdots s_{2d-1}^{-1} s_{2d} = e.$$

Together with  $\rho^k s \neq s$  for  $k = 1, \dots, d-1$ , this shows that  $(s_1, \dots, s_{2d}) \in R_d^d$ . Conversely, if  $(s_1, \dots, s_{2d}) \in R_d^d$ , then  $s$  given by (90) will be in  $R_n^d$ . Hence  $R_n^d$  and  $R_d^d$  have the same number of elements proving that

$$\zeta_n = |R_n| = \sum_{d|n} |R_d^d|.$$

Hence, by the Möbius inversion formula, (cf. Theorem 2.9 in [2]) we have

$$|R_n^n| = \sum_{d|n} \mu\left(\frac{n}{d}\right) \zeta_d.$$

Recall that

$$R_n^n = \{s \in R_n : G_s = H_n = \{id\}\}.$$

Hence the orbit  $G.s$  has  $2n$  elements for all  $s \in R_n^n$ , which implies that  $|R_n^n|$  is divisible by  $2n$  proving (iii).  $\square$

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